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It is denoted/
We write $\lim_{x \to c} f(x) = L$,

which is read "the limit of f of x as x approaches c equals L".

Defn: Existence of a Limit

The limit of a function $f(x)$ exists if and only if the one sided limits of the function are equal.

$\lim_{x \to c} f(x) = L$ if and only if

$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$

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Defn: Continuity:

A function is continuous at 'c' when the following three conditions are hold:

1. $f(c)$ is defined
2. $\lim_{x \to c} f(x)$ exists
3. $\lim_{x \to c} f(x) = f(c)$.

Ex: Limit

Find $\lim_{x \to 1} \frac{x^2-1}{x-1}$

$$= \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)}$$

$$= \lim_{x \to 1} (x+1)$$

$$= 1+1$$

$$= 2.$$

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4.1A Defn:

If $f(x)$ approaches L (where $L \in \mathbb{R}$) as x approaches 'a', if given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad [0 < |x-a| < \delta]$$

It is denoted by,

$$\lim_{x \to a} f(x) = L$$

(or) $f(x) \to L$ as $x \to a$.

Ex: (i) To prove that

$$\lim_{x \to 3} (x^2 + 2x) = 15.$$

Soln: Given, here $f(x) = x^2 + 2x$

and $L = 15$, $a = 3$.

Give $\epsilon > 0$, there exists $\delta > 0$,

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such that,

$$|f(x) - L| < \epsilon \quad [0 < |x-a| < \delta]$$

i.e., $|(x^2+2x) - 15| < \epsilon \quad [0 < |x-3| < \delta]$.

(ii). Show that $\sin\left(\frac{1}{x}\right)$ does not approach a limit as $x \to 0$.

4.1C Theorem:

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then $f(x) + g(x)$ has a limit as $x \to a$ and, in fact,

$$\lim_{x \to a} [f(x) + g(x)] = L + M$$

Proof: Given $\epsilon > 0$, then we must find that $\delta > 0$

Such that,

$$\left| [f(x)+g(x)] - (L+M) \right| < \epsilon, \quad (0 < |x-a| < \delta) \longrightarrow (1)$$

Since $\lim_{x \to a} f(x) = L$,

there exists $\delta_1 > 0$ such that

$$|f(x) - L| < \frac{\epsilon}{2} \quad [0 < |x-a| < \delta_1] \longrightarrow (2)$$

Similarly, there exists $\delta_2 > 0$

Such that,

$$|g(x) - M| < \frac{\epsilon}{2} \quad [0 < |x-a| < \delta_2] \longrightarrow (3)$$

Thus, if $\delta = \min(\delta_1, \delta_2)$ and

if $0 < |x-a| < \delta$, then

$$|f(x) - L| < \frac{\epsilon}{2},$$

$$|g(x) - M| < \frac{\epsilon}{2}$$

and so we have

$$\left| [f(x)+g(x)] - [L+M] \right| = \left| [f(x)-L] + [g(x)-M] \right|$$

$$\leq |f(x)-L| + |g(x)-M|$$

$$\left[\because |x+y| \leq |x| + |y| \right]$$

$$\therefore \left| [f(x)+g(x)] - [L+M] \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad [\because (2) \& (3)]$$

$$\therefore \left| [f(x)+g(x)] - [L+M] \right| < \epsilon$$

Thus equation (1) holds

for $\delta = \min(\delta_1, \delta_2)$.

Hence the proof is complete.

4.1.D Theorem:

If $\lim_{x \to a} f(x) = L$ and

$\lim_{x \to a} g(x) = M$, then

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(a) $\lim_{x\to a}\left[f(x)-g(x)\right] = L-M$,

(b) $\lim_{x\to a} f(x)\,g(x) = L\cdot M$,

and if $M \neq 0$,

(c) $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{L}{M}$.

Proof:

(a) Given $\epsilon > 0$, then we must find that $\delta > 0$ such that

$$\left|\left[f(x)-g(x)\right]-\left[L-M\right]\right| < \epsilon \quad \left[0<|x-a|<\delta\right] \rightarrow ①$$

Since $\lim_{x\to a} f(x) = L$, there exists $\delta_1 > 0$ such that

$$|f(x)-L| < 2\epsilon \quad \left[0<|x-a|<\delta_1\right] \rightarrow ②$$

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Similarly, there exists $\delta_2 > 0$ such that,

$$|g(x)-M| < \epsilon \quad \left[0<|x-a|<\delta_2\right] \rightarrow ③$$

Thus, if $\delta = \min(\delta_1, \delta_2)$ and if $0 < |x-a| < \delta$, then

$$|f(x)-L| < 2\epsilon,$$

$$|g(x)-M| < \epsilon$$

and so we have,

$$\left|\left[f(x)-g(x)\right]-\left[L-M\right]\right|$$

$$= \left|\left[f(x)-L\right]-\left[g(x)-M\right]\right|$$

$$\leq |f(x)-L| - |g(x)-M|$$

$$< 2\epsilon - \epsilon \quad [\because \text{by } ② \,\&\, ③]$$

$$< \epsilon$$

$\therefore \left|[f(x)-g(x)]-[L-M]\right| < \epsilon$

Thus equation ① holds for

$\delta = \min(\delta_1, \delta_2)$.

Hence the proof is complete.

(b) Proof:

Given $\epsilon > 0$, and Assume

that, $|(fg)(x) - LM| = |f(x)g(x) - Mf(x) + Mf(x) - LM|$

$= \left|f(x)[g(x)-M] + M[f(x)-L]\right|$

$\le |f(x)||g(x)-M| + |M||f(x)-L|$ → ①

Since $\lim_{x\to a} f(x) = L$,

therefore, we can choose

$\delta_1 > 0$ such that

$|f(x)-L| < 1 \quad [0<|x-a|<\delta_1]$

$\Rightarrow |f(x)| = |f(x) - L + L|$

$\le |f(x)-L| + |L|$

$|f(x)| \le 1 + |L| \quad [0<|x-a|<\delta_1]$ → ②

Since $\lim_{x\to a} g(x) = M$, then we can choose $\delta_2 > 0$ such that

$|g(x)-M| < \dfrac{\epsilon}{2(1+|L|)} \quad [0<|x-a|<\delta_2]$ → ③

Also, since $\lim_{x\to a} f(x) = L$, then we can choose $\delta_3 > 0$

such that,

$$|f(x) - L| < \cancel{\frac{\epsilon}{2(1+|M|)}} \; \frac{\epsilon}{2|M|}$$

$$[0 < |x-a| < \delta_3] \rightarrow (4)$$

If $\delta = \min\{\delta_1, \delta_2, \delta_3\}$,

then from (1), (2), (3) and (4) we have,

$$|(fg)(x) - LM| \le (1+|L|)\frac{\epsilon}{2(1+|L|)} + |M|\frac{\epsilon}{2|M|}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$\therefore |(fg)(x) - LM| < \epsilon \quad [0 < |x-a| < \delta]$$

Hence $\lim_{x \to a} (fg)(x) = LM$

(C) Proof:

Let $\lim_{x \to a} \left(\frac{f}{g}\right)(x)$

$$= \lim_{x \to a} \left\{ f(x) \cdot \frac{1}{g(x)} \right\}$$

Since $M \neq 0$, then by our known theorem "If $\lim_{x \to a} g(x) = M$, and $M \neq 0$ then $\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M}$"

We have,

$\lim_{x \to a} \frac{1}{g(x)}$ exists and equals $\frac{1}{M}$.

Again, since $\lim_{x \to a} f(x) = L$

$\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M}$.

By our known Theorem

"Let f and g be defined on 'a'. If $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$, then $\lim_{x \to a} (fg)(x) = LM$."

Hence, $\lim_{x \to a} \left\{ f(x) \cdot \frac{1}{g(x)} \right\} = L \cdot \frac{1}{M}$

$= \frac{L}{M}$

$\therefore \lim_{x \to a} \left(\frac{f}{g}\right)(x) = \frac{L}{M}$.

Hence the proof.

4.1E Defn:

Let $f(x)$ approaches L as x approaches infinity if given $\epsilon > 0$ there exists $M \in R$

Such that,

$$|f(x) - L| < \epsilon \quad (x > M)$$

Simply it is written by,

$\lim_{x \to \infty} f(x) = L$, (or)

$f(x) \longrightarrow L$ as $x \longrightarrow \infty$.

Ex: To prove that $\lim_{x \to \infty} \left(\frac{1}{x^2}\right) = 0$.

Soln: Given $\epsilon > 0$, we must find $M \in R$ such that,

$$\left|\frac{1}{x^2} - 0\right| < \epsilon \quad [x > M] \longrightarrow (1)$$

Since (1) is equivalent to

$$\frac{1}{x} < \sqrt{\epsilon} \quad (x > M)$$

it is clear that (1) will hold if to take $M = \frac{1}{\sqrt{\epsilon}}$.