4.1F Defn:

If $f(x)$ approaches L as x approaches 'a' from the right if given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad [a < x < a + \delta]$$

It is denoted as

$$\lim_{x \to a^+} f(x) = L$$

[The number L is called the right-hand limit of f at a]

If $f(x)$ approaches M as x approaches 'a' from the left, if given $\epsilon > 0$

there exists $\delta > 0$ such that

$$|f(x) - M| < \epsilon \quad [a - \delta < x < a]$$

It is denoted as

$$\lim_{x \to a^-} f(x) = M$$

[The number M is called the left-hand limit of f at a]

Note:-

1) If $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$.

consequently, both $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ may exist without being equal to each other.

For example:

If $f(x) = x \quad (0 \le x < 1)$,

$f(x) = 3 - x \quad (1 \le x \le 2)$

(put $x=1$)

$$\left.\begin{aligned} \lim_{x \to 1^-} f(x) &= (1) = 1 \\ \lim_{x \to 1^+} f(x) &= 3 - 1 = 2 \end{aligned}\right\} \neq \text{not equal}$$

4.1 G Defn:

If f is real-valued function on an interval $J \subset R$ is said to be f is nondecreasing on J if

$f(x) \le f(y) \quad [x < y;\ x, y \in J]$

and, we say that f is nonincreasing on J if

$f(x) \ge f(y) \quad [x < y;\ x, y \in J]$

We say that f is monotone if f is either nondecreasing or nonincreasing.

4.1 H Theorem:-

Let f be a nondecreasing function on the bounded open interval (a, b). If f is bounded above on (a, b), then $\lim_{x \to b^-} f(x)$ exists. Also, if f is bounded below on (a, b) then $\lim_{x \to a^+} f(x)$ exists.

Proof:

If f is bounded above and nondecreasing on (a, b).

Let $M = \underset{x \in (a,b)}{\text{l.u.b}}\, f(x)$.

Given $\epsilon > 0$, the number $M - \epsilon$ is thus not an upper bound for the range of f. Hence there exists $y \in (a,b)$ such that

$$f(y) > M - \epsilon.$$

Let $\delta = b - y$.

Then, $f(b-\delta) = f(y) > M - \epsilon$.

Since f is nondecreasing,

$\Rightarrow f(x) > M - \epsilon \quad [b-\delta < x < b]$

Hence, since $f(x) \le M$

for all $x \in (a,b)$, we have

$$|f(x) - M| < \epsilon \quad [b-\delta < x < b]$$

Hence, $\lim_{x \to b^-} f(x) = M$.

If f is bounded below and nondecreasing on (a,b).

$$\text{Let } m = \underset{x \in (a,b)}{\text{g.l.b}} f(x)$$

Given $\epsilon > 0$, the number $m + \epsilon$ is thus not an lower bound for the range of f.

Hence there exists $y \in (a,b)$ such that $f(y) < m + \epsilon$.

~~Let $-\delta = b+y$~~ $\cdot\ -\delta = a+y$ $\quad y \neq a+\delta$

~~Then $f(b+y) = f(y)$~~

$$\text{Let } y = a + \delta$$

$$\Rightarrow f(a+\delta) = f(y) < m + \epsilon$$

Since f is nondecreasing,

$\Rightarrow f(y) < m + \epsilon \quad [a < x < a + \delta]$

Hence, Since $f(x) \leq m$

for all $x \in (a, b)$ we have,

$$|f(x) - m| < \epsilon \quad [a < x < a + \delta]$$

Hence $\lim_{x \to a^+} f(x) = m$.

Hence the proof is complete.

NOTE:-

If f is nonincreasing on (a, b), the following result may be proved by applying 4.1.H to $(-f)$.

4.1 I Theorem:

Let f be a nonincreasing function on the bounded open interval (a, b). If f is bounded above on (a, b), then $\lim_{x \to b^-} f(x)$ exists, while if f is bounded below on (a, b) then $\lim_{x \to a^+} f(x)$ exists.

Proof: First to prove corollary.

4.1 J Corollary:

If is a monotone function on the open interval (a, b), and if $c \in (a, b)$, then $\lim_{x \to c^+} f(x)$ and and

$\lim_{x \to c^-} f(x)$ both exist.

Proof: Suppose that f is nondecreasing. Choose $\delta > 0$ such that $(c-\delta, c+\delta)$ is contained in (a, b).

Then the values of f on the open interval $(c-\delta, c)$ are bounded above by $f(c)$.

By our known Theorem "Let f be a nondecreasing function on the bounded open interval (a, b). If f is bounded above on (a,b), then $\lim_{x \to b^-} f(x)$ exists. Also

if f is bounded below on (a,b) then $\lim_{x \to a^+} f(x)$ exists."

Hence, $\lim_{x \to c^-} f(x)$ exists.

Similarly, the values of f on the open interval $(c, c+\delta)$ are bounded below by $f(c)$, and also by above theorem we have $\lim_{x \to c^+} f(x)$ exists.

Hence proved corollary.

If f is nonincreasing by use "4.1 I" instead of "4.1 H".

Hence proved.