4.1K Defn:

The real-valued function f on the interval $J \subset R$ is said to be strictly increasing if

$$f(x) < f(y) \quad (x < y;\ x, y \in J)$$

similarly, f is said to be strictly decreasing if

$$f(x) > f(y) \quad (x < y;\ x, y \in J)$$

Thus, if f is nonincreasing on J then f is strictly increasing on J if and only if f is $1-1$ on J.

4.2 Metric Space

Refer the book

4.3

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2.2A

Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that S_n approaches the limit L (as n approaches infinity), if for every $\epsilon > 0$ there is a positive integer N such that

$$|S_n - L| < \epsilon \quad (n \geq N)$$

If S_n approaches the limit L we write, $\lim_{n \to \infty} S_n = L$

(or) $S_n \to L \ (n \to \infty)$.

2.3A

If the sequence of real numbers $\{s_n\}_{n=1}^{\infty}$ has the limit L, we say that $\{s_n\}_{n=1}^{\infty}$ is convergent to L.

If $\{s_n\}_{n=1}^{\infty}$ does not have a limit, we say that $\{s_n\}_{n=1}^{\infty}$ is divergent.

Ex: 1) The sequences $1, 1, 1, \ldots$ and $1, \frac{1}{2}, \frac{1}{3}, \ldots$ are convergent.

2) The sequences $1, 2, 3, \ldots$ and $-1, +1, -1, +1, \ldots$ are divergent.

2.10A Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then $\{s_n\}_{n=1}^{\infty}$

is called a Cauchy sequence if for any $\epsilon > 0$ there exists an $N \in I$ such that

$$|s_m - s_n| < \epsilon \quad (m, n \geq I)$$

2.10B If the sequence of real numbers $\{s_n\}_{n=1}^{\infty}$ converges, then $\{s_n\}_{n=1}^{\infty}$ is a cauchy sequence.

4.3C Defn:

4.3E: Let $\langle M, \rho \rangle$ be a metric space. If $\{s_n\}_{n=1}^{\infty}$ is a convergent sequence of points of M, then $\{s_n\}_{n=1}^{\infty}$ is Cauchy.

Proof: Let $L = \lim_{n \to \infty} S_n$

Then, Given $\epsilon > 0$ there exists an $N \in I$ such that,

$\rho(S_m, L) < \epsilon/2$ ~~for every~~

$, \rho(S_n, L) < \epsilon/2$

Thus, if $m, n \geq N$, we have

$\cancel{\rho(S_m, S_n) = \rho(S_m, L)}$.

$|S_m - S_n| = |S_m - L + L - S_n|$

i.e., $\rho(S_m, S_n) = \rho(S_m, L) + \rho(S_n, L)$

$\leq \rho(S_m, L) + \rho(S_n, L)$

$\leq \epsilon/2 + \epsilon/2$

$\rho(S_m, S_n) < \epsilon$

Which proves that $\{S_n\}_{n=1}^{\infty}$ is Cauchy.

5.2 C Theorem:

The real-valued function f is continuous at $a \in R^1$ iff the inverse image under f of any open ball $B[f(a); \epsilon]$ about $f(a)$ contains an open ball $B[a; \delta]$ about a.

i.e., $f^{-1}[B[f(a); \epsilon]] \supset B[a; \delta]$.

Proof:

Let the real valued function f is continuous at $a \in R^1$.

Let us consider that,

$x \in B[a; \delta]$

By our known result

" f is cts. at 'a' iff given $\epsilon > 0$ there exists $\delta > 0$ such that $f(x) \in B[f(a); \epsilon]$ if $x \in B[a; \delta]$."

Since f is cts. at 'a'

iff given $\epsilon > 0$ there exists $\delta > 0$ such that,

$$f(x) \in B[f(a); \epsilon]$$

if $x \in B[a; \delta]$

We have,

$$x \in f^{-1}[B[f(a); \epsilon]]$$

$$\therefore B[a; \delta] \subset f^{-1}(B[f(a); \epsilon])$$

Hence, $f^{-1}([B[f(a); \epsilon]) \supset B[a; \delta]$

Conversely, assume that,

$$f^{-1}(B[f(a); \epsilon]) \supset B[a; \delta]$$

The inverse image under f of any open ball $B[f(a); \epsilon]$ about $f(a)$ contains an open ball $B[a; \delta]$.

Since $f^{-1}[B[f(a); \epsilon]] \supset B[a; \delta]$

Let $x \in f^{-1}(B[f(a); \epsilon])$

We known that by open ball continuous definition,

$$f(x) \in B[f(a); \epsilon]$$

Such that $x \in B[a; \delta]$

Hence, f is continuous at 'a' $\in R$'.

Hence proof.