By our known result

"f is cts. at 'a' iff given $\epsilon > 0$ there exists $\delta > 0$ such that $f(x) \in B[f(a); \epsilon]$ if $x \in B[a; \delta]$."

Since f is cts. at 'a'

iff given $\epsilon > 0$ there exists $\delta > 0$ such that,

$$f(x) \in B[f(a); \epsilon]$$

$$\text{if } x \in B[a; \delta]$$

We have,

$$x \in f^{-1}[B[f(a); \epsilon]]$$

$$\therefore B[a; \delta] \subset f^{-1}(B[f(a); \epsilon])$$

Hence, $f^{-1}([B[f(a); \epsilon]) \supset B[a; \delta]$

Conversely, assume that,

$$f^{-1}(B[f(a); \epsilon]) \supset B[a; \delta]$$

The inverse image under f of any open ball $B[f(a); \epsilon]$ about $f(a)$ contains an open ball $B[a; \delta]$.

Since $f^{-1}[B[f(a); \epsilon]] \supset B[a; \delta]$

Let $x \in f^{-1}(B[f(a); \epsilon])$

We known that by open ball continuous definition,

$$f(x) \in B[f(a); \epsilon]$$

Such that $x \in B[a; \delta]$

Hence, f is continuous at 'a' $\in R$'.

Hence proof.

5.3 Functions continuous on a metric space:

5.3.C Theorem: (10) Mark

The function f is continuous at $a \in M_1$ if and only if any one of the following conditions hold.

(a) Give $\epsilon > 0$ there exists $\delta > 0$ such that

$$\rho_2 [f(x), f(a)] < \epsilon \quad (\rho_1(x,a) < \delta).$$

(b) The inverse image under 'f' of any open ball $B[f(a); \epsilon]$ about $f(a)$ contains an open ball $B[a; \delta]$ about a.

(c) Whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in M_1 converging to a, then the sequence $\{f(x_n)\}_{n=1}^{\infty}$ of points in M_2 converges to $f(a)$.

Proof:

Let the function f is continuous at $a \in M_1$.

~~Now~~ We have to prove the following conditions are equivalent to one another:

(a) $\Rightarrow$ (b) Assume that ~~let~~, given $\epsilon > 0$ there exists $\delta > 0$ such that

$\rho_2[f(x), f(a)] < \epsilon \quad (\rho_1(x, a) < \delta)$.

To prove: (b)

The inverse image under f of any open ball $B[f(a); \epsilon]$ about $f(a)$ contains an open ball $B[a; \delta]$ about a.

Since f is continuous at $a \in M_1$ & it satisfies the conditions,

$\rho_2[f(x), f(a)] < \epsilon \quad (\rho_1(x, a) < \delta)$

By our known theorem (5.2 C)

" The real-valued function f is continuous at $a \in R^1$,

iff given $\epsilon > 0$ $\exists$ $\delta > 0$ such that $f^{-1}[B[f(a); \epsilon]] \supset B[a; \delta]$ "

Hence, the inverse image under 'f' of any open ball $B[f(a); \epsilon]$ about $f(a)$ contains an open ball $B[a; \delta]$ about 'a'.

Hence the proof of (a) $\Rightarrow$ (b).

Next to prove that (b) $\Rightarrow$ (c)

Assume that the inverse image under f of any open ball $B[f(a); \epsilon]$ about $f(a)$ contains an open ball $B[a; \delta]$ about 'a'.