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To prove that (c).

If $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in M_1 converging to a.

Since f is continuous at $a \in M_1$.

By our known Theorem (5.2D)

" The real valued function f is continuous at a iff

$\lim_{n \to \infty} x_n = a \Rightarrow \lim_{n \to \infty} f(x_n) = f(a)$ "

That is, given $\epsilon > 0$, $\exists\ \delta > 0$ such that

$f(x_n) \in B[f(a); \epsilon]$ if $x_n \in B[a; \delta]$

for all $n \geq N$.

Hence, The sequence $\{f(x_n)\}_{n=1}^{\infty}$ of points in M_2 converges to $f(a)$.

Hence the proof of (b) $\Rightarrow$ (c)

Next to prove that (c) $\Rightarrow$ (a)

Assume that $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in M_1 converging to a, then the sequen $\{f(x_n)\}_{n=1}^{\infty}$ of points in M_2 converges to $f(a)$.

To prove that (a).

Since the sequence $\{f(x_n)\}_{n=1}^{\infty}$

of point in M_2 converges to $f(a)$.

By our known theorem 5.2D,
"If f is continuous at a iff $\lim_{n\to\infty} x_n = a \Rightarrow \lim_{n\to\infty} f(x_n) = f(a)$"

so, that f is continuous at 'a' and it satisfies the condition given $\epsilon > 0$, there exists $\delta > 0$ such that

$\rho_2[f(x), f(a)] < \epsilon \quad [\rho_1(x,a) < \delta]$

Hence the proof of (c) $\Rightarrow$ (a)

Conversely,

"Assume that the following three conditions (a), (b) & (c) are hold.

We have to prove that f is continuous at $a \in M_1$.

Since the function f is continuous at $a \in M_1$ if $\lim_{x\to a} f(x) = f(a)$, given $\epsilon > 0$ $\exists$ $\delta > 0$ $\ni$:

$\rho_2[f(x); f(a)] < \epsilon \quad [\rho_1(x,a) < \delta]$

by condition (a) is satisfies.

Moreover, by our known theorem 5.2C "If f is continuous iff given $\epsilon > 0$, $\exists$ $\delta > 0$ such that

$$f^{-1}(B[f(a); \epsilon]) \supset B[a; \delta]"$$

It satisfies the condition (b).

Also, by theorem 5.2.D

" If f is continuous at a iff

$$\lim_{n \to \infty} x_n = a \Rightarrow \lim_{n \to \infty} f(x_n) = f(a)"$$

So, it satisfies the condition (c).

Hence f is continuous at $a \in M$,

Hence the proof of the theorem.