UNIT - V

Taylor's Theorem :

* Taylor's series :

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

when, put $a = 0$ that is,

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

is called the Maclaurin series for f.

8.5.C. Theorem : Taylor formula with the integral form of the remainder

Let f be a real-valued function on $[a, a+h]$ such that $f^{(n+1)}(x)$ exists for every $x \in [a, a+h]$ and $f^{(n+1)}$ is cts on $[a, a+h]$. Then,

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x)$$

where

$$R_{n+1}(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\, dt.$$

The same result holds if $h < 0$ and $[a, a+h]$ is replaced by $[a+h, a]$.

Proof :

$$-R_1(x) = -\int_a^x f'(t)\, dt = f(a) - f(x). \quad \text{Also,}$$

$$R_1(x) - R_2(x) = \frac{f'(a)}{1!}(x-a),$$

$$R_2(x) - R_3(x) = \frac{f''(a)}{2!}(x-a)^2,$$

$$R_n(x) - R_{n+1}(x) = \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

hence add all these equations we obtain,

$$R_{n+1}(x) = R\, f(x) + f(a) + \frac{f'(a)}{1!}(x-a) + \dots + \frac{f^{(n)}}{n!}(x-a)^n.$$

Thus, if f has derivatives of all orders on $[a, a+h]$, and if

$\lim_{n\to\infty} R_{n+1}(x) = 0$, then,

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

Taylor's formula with the lagrange form of the remainder:

8.5.E. Theorem:

Let f be a real-valued function on $[a, a+h]$ such that $f^{(n+1)}(x)$ exists for every $x \in [a, a+h]$ and $f^{(n+1)}$ is cts on $[a, a+h]$. Then if $x \in [a, a+h]$ there exists a number c with $a \leq c \leq x$ such that,

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)$$

$$+ \frac{f^{(n+1)}(c)}{(n+1)}(x-a)^{n+1} \dots$$

The same result holds if $h < 0$ and $[a, a+h]$ is replaced by $[a+h, a]$.

proof:

with $\varphi = f^{(n+1)}$ and $g(t) = (x-t)^n/n!$] we have,

$$R_{n+1}(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt = \frac{f^{(n+1)}(c)}{n!}\int_a^x (x-t)^n\,dt,$$

for some $c \in [a, x]$. Thus.

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

The proof is complete.

Taylor's formula with the cauchy form of the remainder.

8.5.G. Theorem:

Let f be a real-valued function on $[a, a+h]$ such that $f^{(n+1)}(x)$ exists for every $x \in [a, a+h]$ and $f^{(n+1)}$ is cts on $[a, a+h]$. Then if $x \in [a, a+h]$, there exists a number c with $a \le c \le x$ such that,

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-a).$$

The same result holds if $h<0$ and $[a, a+h]$ is replaced by $[a+h, a]$.

Proof:

with $\varphi(t) = f^{(n+1)}(t)(x-t)^n$ and $g(t) = 1$] we have,

$$R_{n+1}(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt$$

$$= \frac{f^{(n+1)}(c)(x-c)^n}{n!}\int_a^x 1\,dt$$

for some $c \in [a, x]$. Thus,

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{n!}(x-c)^n (x-a).$$

The Binomial theorem:

* If $m \in R$ is not a nonnegative integer, then,

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \dots + \frac{m(m-1)\dots(m-n+1)}{n!} + \dots \quad (1)$$

provided that $|x| < 1$.

Proof:

If $f(x) = (1+x)^m$ for $-1 < x < 1$, then,

$$f^{(n)}(x) = m(m-1)\dots(m-n+1)(1+x)^{m-n} \quad (n = 1, 2, \dots).$$

Thus, for any n, Taylor's formula with the cauchy form of the remainder yields,

$$f(h) = 1 + mh + \frac{m(m-1)}{2!}h^2 + \dots + \frac{m(m-1)\dots(m-n+1)}{n!}x^n, \quad (2)$$

where,

$$R_{n+1} = \frac{m(m-1)\dots(m-n)}{n!} \cdot (1+\theta h)^{m-n-1}(1-\theta)^n h^{n+1},$$

$$R_{n+1} = \frac{m(m-1)\dots(m-n)}{n!}\left(\frac{1-\theta}{1+\theta h}\right)^n (1+\theta h)^{m-1} h^{n+1},$$

$$|R_{n+1}| \le \left|\frac{m(m-1)\dots(m-n)}{n!}\right| (1+\theta h)^{m-1} |h|^{n+1}. \quad (3)$$

we emphasize that θ depends on n so that the behaviour of $(1+\theta h)^{m-1}$ as x approches infinity is not obvious, If $m > 1$ then $m-1 > 0$, and so,

$$0 < (1+\theta h)^{m-1} \leq (1+|h|)^{m-1}.$$

If $m < 1$ then,

$$0 < (1+\theta h)^{m-1} = \frac{1}{(1+\theta h)^{1-m}} \leq \frac{1}{(1-|h|)^{1-m}}$$

$$= (1-|h|)^{m-1}.$$

Hence, for any m,

$$(1+\theta h)^{m-1} \leq (1 \pm |h|)^{m-1}.$$

from (3) we, then have,

$$|R_{n+1}| \leq (1 \pm |h|)^{m-1} a_n,$$

where,

$$a_n = \frac{|m(m-1)\dots(m-n)|\,|h|^{n+1}}{n!}.$$

we have thus removed the problem created by θ. Now the ratio test shows that $\sum_{n=1}^{\infty} a_n < \infty$. Hence $\lim_{n \to \infty} a_n = 0$ and so $\lim_{n \to \infty} R_{n+1} = 0$. This and (2) establish (1). The theorem is proved.

8.7. L' hospital rule:

* $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

8.7. A. Theorem:

If $f'(x)$ and $g'(x)$ exist for every x in $(0, \delta]$, if $g'(x) \neq 0$ $(0 < x < \delta)$, if $\lim_{x \to 0^+} f(x) = 0 = \lim_{x \to 0^+} g(x)$ — ①

and if $\lim_{x \to 0^+} \frac{f'(x)}{g'(x)} = L$, — ② then,

$$\lim_{x \to 0^+} \frac{f(x)}{g(x)} = L.$$

proof:

By (1), both f and g will be continuous at 0 if we define $f(0) = 0 = g(0)$. given $x \in (0, \delta]$ there exists $c \in (0, x)$ such that,

$$\frac{f(x) - f(0)}{g(x) - g(0)} = \frac{f'(c)}{g'(c)},$$ where c, of course, depends on x.

Hence, $$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \quad (0 < x < \delta) \quad - ③$$

since c approches 0 as $x \to 0^+$ we have, by ②,

$$\lim_{x \to 0^+} \frac{f'(c)}{g'(c)} = L.$$

Hence the proof.

8.7. D. Theorem:

If $f'(x)$ and $g'(x)$ exist for every x in $(0, \delta]$, if $g'(x) \neq 0$ $(0 < x \leq \delta)$, if $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to 0^+$, and if $\lim_{x \to 0^+} \frac{f'(x)}{g'(x)} = L$, — ①

then, $\lim_{x \to 0^+} \frac{f(x)}{g(x)} = L.$

Proof :

Let $h(x) = f(x) - Lg(x)$ for $0 < x \le \delta$. Then $h'(x) = f'(x) - Lg'(x)$,

and so, by (1),

$$\lim_{x \to 0^+} \frac{h'(x)}{g'(x)} = 0.$$

Given $\epsilon > 0$, this and the hypothesis that $g(x) \to \infty$ as $x \to 0^+$ imply the existence of $\delta_1 > 0$ such that,

$$g(x) > 0 \quad (0 < x \le \delta_1) \quad \text{--- (2)}$$

and such that,

$$\left| \frac{h'(c)}{g'(c)} \right| < \epsilon/2$$

for any $c \in (0, \delta_1)$. If $x \in (0, \delta_1)$, then,

$$\frac{h(\delta_1) - h(x)}{g(\delta_1) - g(x)} = \frac{h'(c)}{g'(c)}$$

for some $c \in (x, \delta_1)$. Hence,

$$\left| \frac{h(x) - h(\delta_1)}{g(x) - g(\delta_1)} \right| < \epsilon/2 \quad (0 < x < \delta_1) \quad \text{--- (3)}$$

since $g(x) \to \infty$ and as $x \to 0^+$, there exists $\delta_2 < \delta_1$ such that,

$$g(x) > g(\delta_1) \qquad (0 < x < \delta_2) \quad \text{--> (4)}$$

from (2) and (4), we thus have,

$$0 < g(x) - g(\delta_1) < g(x) \qquad (0 < x < \delta_2) \quad \text{--- (5)}$$

from (3) and (5) we then conclude

$$\left|\frac{h(x)-h(\delta_1)}{g(x)}\right| < \frac{\epsilon}{2} \quad (0 < x < \delta_2) \quad \text{--(6)}$$

Now, choose $\delta_3 < \delta_2$ such that,

$$\frac{|h(\delta_1)|}{g(x)} < \frac{\epsilon}{2} \quad (0 < x < \delta_3) \quad \text{--(7)}$$

If $0 < x < \delta_2$ we then have,

$$\frac{h(x)}{g(x)} = \frac{h(x)-h(\delta_1)}{g(x)} + \frac{h(\delta_1)}{g(x)},$$

$$\left|\frac{h(x)}{g(x)}\right| \leq \frac{|(h(x)-h(\delta_1)|}{g(x)} + \frac{|h(\delta_1)|}{g(x)},$$

and so, by (6) and (7),

$$\left|\frac{h(x)}{g(x)}\right| < \epsilon \quad (0 < x < \delta_3).$$

This proves $\lim\limits_{x \to 0^+} \frac{h(x)}{g(x)} = 0.$

Since, $\frac{f(x)}{g(x)} = \frac{h(x)}{g(x)} + L.$