UNIT - 4 - CALCULUS

7.1. Set of measure zero :

* f has a Riemaan integral provided f is continious at "almost every point".

Defn :

* The subset E of R^1 is said to be of measure zero if for each $\epsilon > 0$ there exists a finite or countable number of open intervals $I_1, I_2, \ldots$ such that $E \subset \bigcup_n I_n$ and $\sum_n |I_n| < \epsilon$.

Theorem : 7.1.B

If each of the subsets $E_1, E_2, \ldots$ of R^1 is of measure zero, then $\bigcup_{n=1}^{\infty} E_n$ is also of measure zero.

Proof :

Fix $\epsilon > 0$. Since E_n has measure zero, for each $n \in I$ there exists a finite or countable number of open intervals which cover E_n and whose lengths add up to less than $\epsilon/2^n$. The union of all such open intervals (for all $n \in I$) then covers $\bigcup_{n=1}^{\infty} E_n$, and the lengths of all these (countably many*) intervals add up to $< \epsilon/2 + \epsilon/2^2 + \cdots = \epsilon$

Hence $\bigcup_{n=1}^{\infty} E_n$ has measure zero.

Corollary :

Every countable subset of R^1 has measure zero.

Cantor set :

* There are even uncountable subsets of R' which have measure zero.

7.1. D. Defn :

* A statement is said to hold at almost every point of $[a,b]$ (or almost everywhere in $[a,b]$) if the set of points of $[a,b]$ at which the statement does not hold is of measure zero.

* "f is continuous at almost every point of $[a,b]$" means the same as "if E is the set of points of $[a,b]$ at which f is not continous, then E is of measure zero." we could also say "f is continuous almost everywhere in $[a,b]$."

7.2. Defn of the Riemann Integral :

* Let J be any bounded interval of real numbers, and let f be a bounded (real-valued) function on J. we define $M[f:J]$, $m[f:J]$ and $w[f:J]$.

$$M[f;J] = \text{l.u.b.} f(x) \quad x \in J$$

$$m[f:J] = \text{g.l.b} f(x) \quad x \in J$$

$$w[f;J] = M[f:J] - m[f;J].$$

(Thus, $w[f;J]$ is exactly the same as in definition save that we now do not require that J be open). If a is a point of J, we define $w[f;a]$ as

$$w[f;a] = \text{g.l.b} \; w[f;J]$$

where the g.l.b is taken over all open subintervals J of $\mathcal{J}$ such that $a \in J$.* (This is also consistent with 5.6 B.).

7.2. Defn :

* By a subdivision of the closed bounded interval $[a,b]$ we mean a finite subset $\{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ such that $a = x_0 < x_1 < \ldots < x_n = b$. If R , for example, that $[0, \frac{1}{2}]$ is an open subinterval of $[0,1]$.

* σ and τ are two subdivisions of $[a,b]$. we say that τ is a refinement of σ if $\sigma \subset \tau$. That is, τ is a refinement of σ means that the subdivision of τ is obtained from the subdivision σ by adding more" points of subdivision

* If $\sigma = \{x_0, x_1, \ldots, x_n\}$ is a subdivision of $[a,b]$, then the closed intervals $I_1 = [x_0, x_1]$, $I_2 = [x_1, x_2]$ ... $I_n = [x_{n-1}, x_n]$ are called the component intervals of σ.

7.2. C. Defn :

* Let f be a bounded function on the closed bounded interval $[a,b]$ and let σ be subdivision of $[a,b]$. we define $U[f;\sigma]$, called the upper sum for, corresponding to σ as

$$U[f;\sigma] = \sum_{k=1}^{n} M[f; I_k] \cdot |I_k|.$$

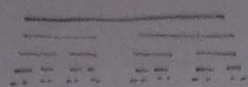
Then $I_1, \ldots I_n$ are the component intervals of σ. Illy the lower sum $L[f;\sigma]$ define as $\{U[F;\sigma] \geq L[F;\sigma]\}$

$$L[f;\sigma] = \sum_{k=1}^{n} m[f; I_k] \cdot |I_k|.$$

1. Cantor set :

* Cantor set is uncountable and has measure zero.

Ex:



* Delete the middle third
* Do the same for each remaining segment.
* Repeat this for ever leaving the cantor set.
* All length is gone, but an infinity of point remain.

2. Bounded intervals of real numbers :

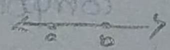
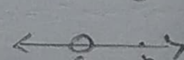
* An interval is said to be bounded if both of its end points are real numbers.

Ex: The interval (1,10) is considered bounded.

(-∞, ∞) is unbounded.

3. Interval Notations :

* Notation Type ; Inequality Graph Notation.

[a,b]	closed	$a \le x \le b$	
(a,b)	open	$a < x < b$	
[a,b]	Half open	$a \le x < b$	
(a,b]	Half open	$a < x \le b$	

4. Bounded above :

* If f is real valued and $f(x) \le A \ \forall \ x \in X$. Then the function said to be bounded above by A.

5. Bounded below :

* If f is real valued and $f(x) \ge B \ \forall \ x \in X$. Then the function said to be bounded below by B.

6. How to find upper bound and lower bound?

* Half the degree of accuracy specified.
* Add to get the upper bound.
* Subtract to get the lower bound.

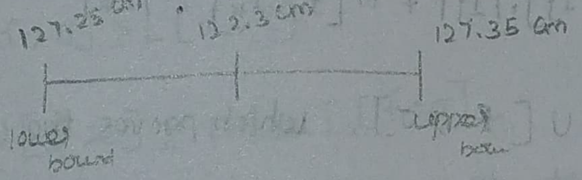
Ex. 1. Find the l.b and U.b 450 to the nearest 10.

Soln:

* Half the degree accuracy = $10 \div 2 = 5$
* Upper bound $= 450 + 5 = 455$.
* Lower bound $= 450 - 5 = 445$.

Ex. 2. The length is given 127.3 cm. correct to one decimal places.

Soln:



T.2. D. LEMMA:

* Let f be a bounded function on $[a, b]$. Then every upper sum for f is greater than or equal to every lower sum for f. That is, if σ and τ are any two subdivisions of $[a, b]$, then $U[f;\sigma] \geq L[f;\tau]$.

Proof:

We shall first show that if σ^* is any refinement of σ then

$$U[f;\sigma] \geq U[f;\sigma^*]. \quad \text{--- ①}$$

It is enough to prove this in the case where σ^* is obtained from σ by adding only one more point of subdivision.

$\sigma \Rightarrow \sigma^* \supset \sigma$

$\sigma = I_1, \dots, I_k, \dots, I_n.$

$\sigma^* = I_1, \dots, I_k^*, I_k^{**}, \dots, I_n.$

where $|I_k| = |I_k^*| + |I_k^{**}|$ and $I_k = I_k^* \cup I_k^{**}$

since, $I_k^* \subset I_k$, $I_k^{**} \subset I_k$. Then

$M[f; I_k^*] \leq M[f; I_k]$.

$M[f; I_k^{**}] \leq M[f; I_k]$

$$\therefore U[f;\sigma^*] = \sum_{j=1}^{n} M[f;I_j]\cdot|I_j| + M[f;I_k^*]\cdot|I_k^*| + M[f;I_k^{**}]\,|I_k^{**}|$$

$$\leq \sum_{j=1}^{n} M[f;I_j]\cdot|I_j| + M[f;I_k]\cdot|I_k^*|+|I_k^{**}|$$

$$\leq \sum_{j=1}^{n} M[f;I_j]\cdot|I_j| + M[f;I_k]\cdot|I_k|$$

$U[f;\sigma^*] \leq U[f;\sigma]$. which proves the condition ①.

similarly,

If τ^* is any refinement of τ it may be shown that

$$L[f;\tau^*] \geq L[f;\tau]. \quad \text{--- ②}$$

$\tau \Rightarrow \tau^* \supset \tau$

$\tau = I_1, \dots, I_l, \dots I_n$

$\tau^* = I_1, \dots, I_l^*, I_l^{**}, \dots, I_n$

where $|I_l| = |I_l^*| + |I_l^{**}|$ and $I_l = I_l^* \cup I_l^{**}$

$I_l^* \subset I_l$, $I_l^{**} \subset I_l$

$m[f; I_l^*] \geq m[f; I_l]$

$m[f; I_l^{**}] \geq m[f; I_l]$

$L[f;T^*] \geq L[f;T]$

$$L[f;T^*] = \sum_{j=1}^{n} m[f;I_j]\cdot|I_j| + m[f;I_i^*]\cdot|I_i^*| + m[f;I_i^{**}]\cdot|I_i^{**}|$$

$$\geq \sum_{j=1}^{n} m[f;I_j]\cdot|I_j| + m[f;I_i^* \cup I_i^{**}]\cdot(|I_i^*| + |I_i^{**}|)$$

$$\geq \sum_{j=1}^{n} m[f;I_j]\cdot|I_j| + m[f;I_i]\cdot|I_i|$$

$$L[f;T^*] \geq L[f;T] \quad \text{---} ②$$

eqn ① ②, we prove that,

$$U[f;\sigma] \geq U[f;\sigma^*] \geq L[f;T^*] \geq L[f;T].$$

$$U[f;\sigma] \geq L[f;T].$$

Relation between upper and lower integrals:

* $U[f;\sigma] \geq L[f;T]$

$$\text{g.l.b } U[f;\sigma] \geq \text{l.u.b. } L[f;\sigma].$$

where the g.l.b. and l.u.b are both taken over all subdivisions σ of $[a,b]$. For if T is any subdivision of $[a,b]$, then the lemma shows that $L[f;T]$ is a lower bound for the set of all upper sums $U[f;\sigma]$.

Hence,

$$L[f;T] \leq \text{g.l.b. } U[f;\sigma],$$

for every subdivision T. But this says that g.l.b. $U[f;\sigma]$ is an upper bound for the set of all lower sums $L[f;T]$.

Hence,

$$\text{l.u.b. } L[f;T] \leq \text{g.l.b } U[f;\sigma]$$

which is equivalent to (1). The inequality (1) gives us an important relation between the upper and lower integrals of a function.

Defn : Upper and lower integral :

* Let f be a bounded function on the closed bounded interval $[a,b]$, we define,

$\overline{\int_a^b} f(x)\,dx$, called the upper integral of f over $[a,b]$ as,

$$\overline{\int_a^b} f(x)\,dx = \text{g.l.b } U[f;\sigma].$$

where the g.l.b taken over all subdivisions σ of $[a,b]$.

Similarly, we define,

$\underline{\int_a^b} f(x)\,dx$, called the lower integral of f over $[a,b]$ as

$$\underline{\int_a^b} f(x)\,dx = \text{l.u.b. } L[f;\sigma].$$

we sometime denote, $\overline{\int_a^b} f$ and $\underline{\int_a^b} f$. --- ①

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f. \quad \text{- ②} \quad \text{from inequality ①}$$

we shall presently show that for continious function f,

$\underline{\int_a^b} f$ and $\overline{\int_a^b} f$ are equal, there exists f for which

$$\underline{\int_a^b} f < \overline{\int_a^b} f.$$

7.2, F. Defn.

* If f is bounded function on the closed bounded interval $[a,b]$ we say that f is Riemann integrable on $[a,b]$ if

$$\underline{\int_a^b} f = \overline{\int_a^b} f.$$

In this case we define $\int_a^b f(x)\,dx$ (or $\int_a^b f$) as

$$\int_a^b f = \int_{-a}^{b} f = \int_a^{-b} f.$$

we denone by $R[a,b]$ the class of all functions f which are Riemann integrable on $[a,b]$.

7.2.G. Theorem :

* Let f be a bounded function on the closed bounded interval $[a,b]$. Then $f \in R[a,b]$ if and only if, for each $\epsilon > 0$, there exists a subdivision σ of $[a,b]$ such that,

$$U[f;\sigma] < L[f;\sigma] + \epsilon \quad \text{--- (1)}$$

proof :

suppose first that given $\epsilon > 0$ there exists σ such that (1) holds. Then, since

$$\int_a^{-b} f \leq U[f;\sigma] \text{ and } \int_{-a}^{b} f \geq L[f;\sigma],$$

we have, $\int_a^{-b} f < \int_{-a}^{b} f + \epsilon$. since ϵ was arbitary, it follows that,

$$\int_a^{-b} f \leq \int_{-a}^{b} f,$$ and hence, by (2) of 7.2.E, that

$$\int_{-a}^{b} f = \int_a^{-b} f.$$ This proves $f \in R[a,b]$.

conversely, suppose $f \in R[a,b]$. Then,

$$\int_a^{-b} f = \text{g.l.b. } U[f;\sigma] = \text{l.u.b. } L[f;\tau] = \int_{-a}^{b} f.$$

Given $\epsilon > 0$ we may (by defn of g.l.b) choose a subdivision σ such that,

$$\int_a^b f + \epsilon/2 > U[f;\sigma]$$

Similarly we may choose a subdivision τ such that,

$$\int_a^b f - \epsilon/2 < L[f;\tau].$$ Hence,

$$L[f;\tau] + \epsilon/2 > U[f;\sigma] - \epsilon/2.$$

By ① and ② of 7.2 D, we then have,

$$L[f;\sigma \cup \tau] + \epsilon/2 > U[f;\sigma \cup \tau] - \epsilon/2.$$

This is equivalent to (1) (with $\sigma \cup \tau$ in place of σ).

7.4.A. Theorem:

* If $f \in R[a,b]$ and $a < c < b$, then $f \in R[a,c]$, $f \in R[c,b]$

then $$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof:

By 7.3.A. the set E of points in $[a,b]$ at which f is not continious set of measure zero. obviously, then, $E \cap [a,c]$ is of measure zero and so $f \in R[a,c]$ III^{ly}, $f \in R[c,b]$.

If σ is any subdivision of $[a,c]$ and τ is any subdivision of $[c,b]$, then $\sigma \cup \tau$ is any subdivision of $[a,b]$ whose component intervals are those of σ together with set of τ

Hence,
$$L[f;\sigma] + L[f;\tau] = L[f;\sigma \cup \tau] \leq \int_a^b f.$$

and so, $$L[f;\sigma] + L[f;\tau] \leq \int_a^b f.$$

By taking the least upper bound on the left over all σ, we obtain
$$\int_a^b f + L[f;\tau] \leq \int_a^b f.$$

Now taking the least upper bound over all τ we have,

$$\int_a^c f + \int_c^b f \le \int_a^b f.$$

Going last to the original σ and τ we also have

$$U[f;\sigma] + U[f;\tau] = U[f;\sigma\cup\tau] \ge \int_a^{-b} f,$$ so that

$$U[f;\sigma] + U[f;\tau] \ge \int_a^b f.$$

Taking greatest lower bounds as in the first part of the proof we obtain $\int_a^c f + \int_c^b f \ge \int_a^b f$. The theorem follows from (1) and (2).

7.4.B. Theorem:

* If $f \in R[a,b]$ and λ is any real number, then $\lambda f \in R[a,b]$ and $\int_a^b \lambda f = \lambda \int_a^b f$.

Proof:

If $\lambda = 0$ the theorem is obvious. Suppose $\lambda > 0$. Since λf is a continuous at every point where f is continuous, it is clear that $\lambda f \in R[a,b]$. Since $\lambda > 0$, if J is any interval contained in $[a,b]$ then,

$M[\lambda f;J] = \lambda M[f;J]$. and so, for any subdivision σ of $[a,b]$,

$$U[\lambda f;\sigma] = \lambda U[f;\sigma].$$

It follows easily, on taking the g.l.b of both sides that

$$\int_a^b \lambda f = \lambda \int_a^b f. \quad (\lambda > 0) \quad \text{--- (1)}$$

Hence the theorem proved for $\lambda > 0$.

Now for any J we also have,

$$M[-f; J] = -m[f; J]. \quad \text{Hence,}$$

$$\int_a^b (-f) = \text{g.l.b.}\, U[-f;\sigma] = \text{g.l.b.}\{-L[f;\sigma]\} = -\text{l.u.b.}\, L[f;\sigma] = -\int_a^b f$$

That is,

$$\int_a^b (-f) = -\int_a^b f. \quad \text{--- (2)}$$

If $\mu < 0$ then $\lambda = -\mu > 0$ and so, by (2) and (1) (since $\lambda f \in \mathcal{R}[a,b]$)

$$\int_a^b \mu f = \int_a^b (-(\lambda f)) = -\int_a^b \lambda f = -\lambda \int_a^b f = \mu \int_a^b f.$$

This completes the proof.

7.4.C Theorem:

If $f \in \mathcal{R}[a,b]$, $g \in \mathcal{R}[a,b]$, then $f+g \in \mathcal{R}[a,b]$ and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

Proof:

By T.3 A, the sets E_f and E_g of points at which f and g respectively, are not continous are both of measure zero. Hence, by 7.1.B, the set $E_f \cup E_g$ is of measure zero.

But if $x \in [a,b] - (E_f \cup E_g)$, then f, g and hence $f+g$ are continous at x. Thus $f+g$ is continous at almost every point in $[a,b]$, and so $f+g \in \mathcal{R}[a,b]$

If J is any interval contained in $[a,b]$, and if $y \in J$,

we have $f(y) + g(y) \le M[f;J] + M[g;J]$.

Hence $M[f+g;J] \le M[f;J] + M[g;J]$

For any subdivision σ, then we have,

$$\int_a^b (f+g) \le \cancel{U[f+g]}\ U[f+g;\sigma] \le U[f;\sigma] + U[g;\sigma] \quad ①$$

But gn $\epsilon > 0$ there is a subdivision σ_1 of $[a,b]$ such that,

$$U[f;\sigma_1] < L[f;\sigma_1] + \frac{\epsilon}{2} \le \int_a^b f + \epsilon/2.$$

Also, there is a subdivision σ_2 of $[a,b]$ such that,

$$U[g;\sigma_2] < L[g;\sigma_2] + \epsilon/2 \le \int_a^b g + \epsilon/2.$$

If $\sigma = \sigma_1 \cup \sigma_2$ then by ①,

$$U[f;\sigma] < \int_a^b f + \epsilon/2, \quad U[g;\sigma] < \int_a^b g + \epsilon/2$$

From ① we the have,

$$\int_a^b (f+g) < \int_a^b f + \int_a^b g + \epsilon$$

Since ϵ was abitary, this proves,

$$\int_a^b (f+g) \le \int_a^b f + \int_a^b g. \quad --②$$

Since f and g were any Riemann integrable functions in ②, Hence, we can substitute $-f, -g$ for f, g

$$\int_a^b (-f-g) \le \int_a^b (-f) + \int_a^b (-g). \quad \text{(Then we have using 7.4 B.}$$

$$-\int_a^b (f+g) \le -\left(\int_a^b f + \int_a^b g\right) \quad --- ③$$

Now, multiply both sides of (3) by -1. This reverse the ineq

and so,

$$\int_{a}^{b} (f+g) \geq \int_{a}^{b} f + \int_{a}^{b} g \quad \text{---} > (4)$$

The theorem follows from (2) and (4).

7.4. D. Lemma :

If $f \in R[a,b]$ and if $f(x) \geq 0$ almost every where $(a \leq x \leq b)$, then $\int_{a}^{b} f \geq 0$.

7.4. E. Corallary :

If $f \in R[a,b]$, $g \in R[a,b]$ and if $f(x) \leq g(x)$ almost every where $(a \leq x \leq b)$, then $\int_{a}^{b} f \leq \int_{a}^{b} g$.

proof :

By 7.4 B and 7.4 C the function $-f$ and $g-f$ are Riemann integrable. Since $g(x) - f(x) \geq 0$ almost everywhere we have, by 7.4 D, 7.4 C and 7.4 B.

$$0 \leq \int_{a}^{b} (g-f) = \int_{a}^{b} [g + (-f)] = \int_{a}^{b} g + \int_{a}^{b} (-f) = \int_{a}^{b} g - \int_{a}^{b} f.$$

This proves the corralary.

7.4. F. Corollary :

If $f \in R[a,b]$, then $|f| \in R[a,b]$ and $\left| \int_{a}^{b} f \right| \leq \int_{a}^{b} |f|$.

proof :

Since $|f|$ will be continuous at every point where f is continuous, it is clear by 7.3 A that $|f| \in \mathcal{R}[a,b]$.

Now, since $f(x) \leq |(f(x))| = |f|(x) \ \forall \ x \in [a,b]$.

7.4.E implies

$$\int_a^b f \leq \int_a^b |f| \quad \text{--- (1)}$$

since $-f(x) \leq |f(x)| \ \forall \ x \in [a,b]$, 7.4.E $\Rightarrow$

$$-\int_a^b f \leq \int_a^b |f| \quad \text{--- (2)}$$

The corollary follows from (1) and (2).

7.4.G.

If $b < a$, we define $\int_a^b f$ to be $-\int_{-b}^{-a} f$,

provided that $f \in \mathcal{R}[b,a]$. It is then not difficult to

show that result such as

$$\int_a^c f + \int_c^b f = \int_a^b f.$$ hold, regardless of the

order of the points a, b, c.

Rolle's Theorem:

* Let f be function defined on $[a,b]$ such that,

(i) f is cts on $[a,b]$;

(ii) f is derivable on $]a,b[$;

(iii) $f(a) = f(b)$.

Then there exists a real-number c between a and b

Such that $f'(c) = 0$.

Proof:

Since f is cts on $[a,b]$, and since every function that is cts on a closed interval is bounded theorem, therefore, f must be bounded on $[a,b]$. Let $\sup f = M$, $\inf f = m$.

Two different cases arise:

(1) $M = m$. Then f is constant over $[a,b]$ and consequently, $f'(x) = 0$, for all x in $[a,b]$.

(2) $M \neq m$. Since $f(a) = f(b)$, therefore, at least one of the numbers M and m is different from $f(a)$ and therefore, also from $f(b)$. For the sake of definiteness, assume that $M \neq f(a)$.

Since every function that is cts on a closed interval attains its supremum, therefore, there exists some real number c in $[a,b]$. Such that $f(c) = M$. Further, since $f(a) \neq M \neq f(b)$, Therefore, c is different from both a and b. This means that c lies in the open interval $]a,b[$.

Since $f(c)$ is the supremum of f of on $[a,b]$, therefore

$$f(x) \leq f(c) \quad \text{for all } x \text{ in } [a,b] \qquad --(1)$$

In particular,

$$f(c-h) \leq f(c), \text{ for all positive real numbers } h$$

such that $c-h$ lies in $[a,b]$. This means that,

$$\frac{f(c-h) - f(c)}{-h} \geq 0.$$

for all positive real numbers h such that $c-h$ lies in $[a,b]$, Taking limits as $h \to 0$ and observing that since $f'(x)$ exists at each point $]a,b[$. and therefore, in particular at $x=c$, we have

$$Lf'(c) \geq 0 \quad \text{-- (ii)}$$

from (i), we similarly have,

$$f(c+h) \leq f(c).$$

for all positive real numbers h such that $c+h$ lies in $[a,b]$. By the same argument as above we have

$$Rf'(c) \leq 0 \quad \text{--- (iii)}$$

since $f'(x)$ exists at $x=c$. Therefore,

$$Lf'(c) = f'(c) = Rf'(c) \quad \text{--> (iv)}$$

from (ii), (iii), (iv), we find that $f'(c) = 0$,

The case $M = f(a) \neq m$ can be disposed of in the same manner as above.