UNIT - III

Continious functions on metric Spaces

5.1 A.

1. If continious fn is differentiable?

(i) differentiable fn must be cts at every point in its

(ii) cts fn need not be differentiable.

(iii) The limit of f as x approches a is equal to f (a).

Defn:

A fn f is cts at a point $x = a$ when,

(i) a fn is defined at a.

(ii) The limit of f as x approches a from the right hand and left hand limit exist and are equal.

5.1. B. Theorem:

* If the real - valued functions f and g are continous at $a \in R^1$, then so are $f+g$, $f-g$, and fg. If $g(a) \neq 0$, then f/g is also continous at a.

Proof:

Since f and g are continuous at a we have,

$$\lim_{x \to a} f(x) = f(a), \quad \lim_{x \to a} g(x) = g(a).$$

$$\lim_{x \to a} [f(x) + g(x)] = f(a) + g(a).$$

$$\lim_{x \to a} (f+g)(x) = (f+g)(a).$$

This proves that $f+g$ is continious at a.

A continious function of a continious function is continious.

5:1.C Theorem:

* If f and g are real-valued functions, if f is continious at a, and if g is continious at $f(a)$, then $g \circ f$ is continious at a.

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proof:

we must show $\lim_{x \to a} g \circ f(x) = g \circ f(a)$ or

$$\lim_{x \to a} g[f(x)] = g[f(a)].$$

That is, given $\epsilon > 0$ we must find $\delta > 0$ such that,

$$|g[f(x)] - g[f(a)]| < \epsilon \quad (0 < |x-a| < \delta) \quad \text{--- (1)}$$

Let $b = f(a)$. Now by hopothesis $\lim_{y \to b} g(y) = g(b)$. Hence there exists $\eta > 0$ such that,

$$|g(y) - g(b)| < \epsilon \quad (0 < |y-b| < \eta) \quad \text{--- (2)}$$

$$\lim_{x \to a} f(x) = f(a)$$

Thus (using η where we usually use ϵ) there exists δ such that,

$$|f(x) - f(a)| < \eta \quad (|x-a| < \delta) \quad (\text{or})$$

$$|f(x) - b| < \eta \quad (|x-a| < \delta) \quad \text{--- (3)}$$

Thus if $|x-a| < \delta$ then $f(x)$ is within η of b and so we may substitute $f(x)$ for y in (1). Hence,

$$|g[f(x)] - g(b)| < \epsilon \quad (|x-a| < \delta).$$

The proof is complete.

5.2A. Theorem :

* The real-valued function f is continious at $a \in R'$ if and only if given $\epsilon > 0$ there exists $\delta > 0$ such that,

$$|f(x) - f(a)| < \epsilon \quad (|x - a| < \delta).$$

5.2.B. Defn :

* If $a \in R'$ and $r > 0$ we define $B[a;r]$ to be the set of all $x \in R'$ whose distance to a is less than r. That is

$$B[a;r] = \{x \in R' \mid |x-a| < r\}$$

we call $B[a;r]$ the open ball of radius r about a.

Note :

"f is continuous at a if and only if given $\epsilon > 0$ there is an open ball exists $\delta > 0$ such that $f(x) \in B[f(a);\epsilon]$ if $x \in B[a;\delta]$".

5.2.C. Theorem :

* The real-valued function f is continious at $a \in R'$ if and only if the inverse image* under f of any open ball $B[f(a);\epsilon]$ about $f(a)$ contains an open ball $B[a;\delta]$ about a. (That is given $\epsilon > 0$ there exists $\delta > 0$ such that $f^{-1}(B[f(a);\epsilon]) \supset B[a;\delta]$.

proof :

Let the real-valued function f is continous at $a \in R'$.

Let us consider that $x \in B[a;\delta]$

By our known result "f is cts. at 'a' iff gn $\epsilon > 0$ there exists $\delta > 0$ such that $f(x) \in B[f(a);\epsilon]$ if $x \in B[a;\delta]$".

Since f is cts. at 'a' iff $\forall$ $\epsilon > 0$ there exists $\delta > 0$ such that, $f(x) \in B[f(a); \epsilon]$ if $x \in B[a; \delta]$. we have,

$x \in f^{-1}[B[f(a); \epsilon]]$.

$\therefore B[a; \delta] \subset f^{-1}(B[f(a); \epsilon])$. Hence,

$f^{-1}([B(f(a); \epsilon]) \supset B[a; \delta]$

conversely, assume that, $f^{-1}(B[f(a); \epsilon]) \supset B[a; \delta]$

The inverse image of under f of any open ball $B[f(a); \epsilon]$ about $f(a)$ contains an open ball $B[a; \delta]$.

Since $f^{-1}[B[f(a); \epsilon]] \supset B[a; \delta]$.

Let $x \in f^{-1}(B[f(a); \epsilon)$. we known that by open ball continues definition,

$f(x) \in B[f(a); \epsilon]$. Such that $x \in B[a; \delta]$.

Hence, f is continous at 'a' $\in R'$.

Hence Proof.

Note:

The sequence $\{x_n\}_{n=1}^{\infty}$ converges to a if and only if given $\epsilon > 0$ there exists $N \in I$ such that,

$$x_n \in B[a; \epsilon] \quad (n \geq N).$$

5.2.D Theorem:

The real-valued function f is continous at $a \in R^1$ if and only if, whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers converging to a, then the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(a)$. That is, f is continous at a if and only if,

$$\lim_{n \to \infty} x_n = a \rightarrow \lim_{n \to \infty} f(x_n) = f(a) \quad \text{--} (*)$$

Proof:

Let us first assume that f is continous at a and P.T $(*)$ holds. Let $\{x_n\}_{n=1}^{\infty}$ be any sequence of real numbers converging to a. [Then $f(x_n)$ will be defined for n sufficiently large]. we must show that $\lim_{n \to a} f(x_n) = f(a)$ - that is given $\epsilon > 0$, we must find $N \in I$ such that,

$$f(x_n) \in B[f(a); \epsilon] \quad (n \geq N). \quad \text{-- } 1$$

but since f is cts at a there exists $\delta > 0$ such that,

$$f(x) \in B[f(a); \epsilon] \quad (x \in B[a, \delta]) \quad \text{-- } 2$$

Since $\lim_{n \to \infty} x_n = a$, there exists $N \in I$ such that,

$$x_n \in B[a; \delta] \quad (n \geq N) \quad \text{--} 3$$

Conversely, suppose $(*)$ holds. We must prove that f is cts at a. Assume use contrary. Then, for some $\epsilon > 0$, the inverse image under f of $B = [f(a); \epsilon]$ contains no open ball about a. In particular, $f^{-1}(B)$ does not contain

$B[a; 1/n]$ for any $n \in I$. Thus, for each $n \in I$, there is a pt, $x_n \in B[a; 1/n]$ such that $f(x_n) \notin B$. That is,

$|x_n - a| < 1/n$ but $|f(x_n) - f(a)| \geq \epsilon$.

This clearly contradicts <*>, so f must be cts at a.

Note : $\lim_{n \to \infty} g[f(x_n)] = g[f(a)]$. --> (1)

where $\{x_n\}_{n=1}^{\infty}$ is any sequence of real numbers such that,

$\lim_{n \to \infty} x_n = a$. --> (2)

Since f is cts at a, (2), and 5.2 D imply

$\lim_{x \to \infty} f(x_n) = f(a)$. --> (3)

But ∵ g is cts at $f(a)$, (3), 5.2 D imply (1) and the proof is complete.

5.3. Functions continous on a metric space.

5.3 A. Defn : 3

* Let $\langle M, \rho \rangle$ be a metric space. If $a \in M$ and $r > 0$, then, $B[a; r]$ is defined to be the set of all points in M whose distance to a is less than r. That is,

$$B[a; r] = \{x \in M \mid \rho(x, a) < r\}.$$

we call $B[a, r]$ the open ball of radius r about a.

* the open ball of radius 1 about the origin in Euclidean 3-space in the set of all points $\langle x, y, z \rangle$ such that $x^2 + y^2 + z^2 < 1$.

5.3. B. Defn:

* The function f is continious at $a \in M$, if $\lim_{x \to a} f(x) - f(a)$

5.3C. Theorem.

The function f is continious at $a \in M$, if and only if any one of the following conditions hold.

(a) Given $\epsilon > 0$ there exists $\delta > 0$ such that $\rho_2 [f(x), f(a)] < \epsilon$ $(\rho_1 (x, a) < \delta)$

(b) The inverse image under 'f' of any open ball $B[f(a); \epsilon]$ about $f(a)$ contains an open ball $B[a; \delta]$ about 'a'.

(c) whenever $\{x_n\}^{\infty}$ is a sequence of points in M, converging to a, then the sequence $\{f(x_n)\}_{n=1}^{\infty}$ of points in M_2 converges to $f(a)$.

proof:

Let the function f is continous at a $a \in M$, we have to prove the following conditions are equivalent to one another:

(a) => (b) assume that given $\epsilon > 0$, there exists $\delta > 0$ such that $\rho_2 [f(x), f(a)] < \epsilon$ $(\rho_1 (x, a) < \delta)$.

TO prove: (b)

The inverse image under f of any open ball $B[f(a); \epsilon]$ about $f(a)$ contains an open ball $B[a; \delta]$ about a.

since f is continous at $a \in M_1$, and it satisfies the conditions.

$\rho_2 [f(x), f(a)] < \epsilon$ $(\rho_1 (x, a) < \delta)$

By our known theorem (5.2 C). "The real-valued function f is continous at $a \in R'$ iff given $\epsilon > 0$ $\exists$ $\delta > 0$ such that $f^{-1} [B[f(a); \epsilon] \supset B[a; \delta]$. Hence, the inverse image under 'f' of any open ball $B[f(a); \epsilon]$ about $f(a)$ contains an open ball $B[a; \delta]$ about 'a'. Hence the proof of (a) => (b).

Next to prove that (b) => (c)

* Assume that the inverse image under f of any open ball $B[f(a); \epsilon]$ about $f(a)$ contains an open ball $B[a; \delta]$ about 'a'.

To prove that (c).

If $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in M, converging to a.

since f is continuous at $a \in M$.

By our known Theorem (5.2 D). " The real-valued function f is contious at a iff $\lim_{n \to \infty} x_n = a \Rightarrow \lim_{n \to \infty} f(x_n) = f(a)$".

That is, given $\epsilon > 0$, $\exists$ $\delta > 0$ such that,

$f(x_n) \in B[f(a); \epsilon]$ if $x_n \in B[a; \delta]$ $\forall$ $n \geq N$.

Hence, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ of points in M_2 converges to $f(a)$. Hence the proof of (b) => (c).

Next to prove that (c) => (a)

Assume that $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in M_1 converging to a, then the sequence $\{f(x_n)\}_{n=1}^{\infty}$ of points in M_2 converges to $f(a)$.

To prove that (a):

since the sequence $\{f(x_n)\}_{n=1}^{\infty}$ of point in M_2 converges to $f(a)$

By our known theorem 5.2D, "If f is continous at a iff $\lim_{n\to\infty} x_n = a \Rightarrow \lim_{n\to\infty} f(x_n) = f(a)$". So that f is cts at 'a' and it satisfies the condition given $\epsilon > 0$, there exists $\delta > 0$ such that, $\rho_2[f(x), f(a)] < \epsilon$ $[\rho_1(x,a) < \delta]$

conversely,

Assume that the following three conditions (a), (b) & (c) are hold. we have to prove that f is continous at $a \in M_1$.

since the function f is cts at $a \in M_1$ if,

$\lim_{x\to a} f(x) = f(a)$ ε, given $\epsilon > 0$ $\exists\ \delta > 0 \ni$: $\rho_2[f(x); f(a)] < \epsilon$. $[\rho_1(x,a) < \delta]$. by condition (a) is satisfies. Moreover, by our known theorem 5.2.C "If f is cts iff $g_n\ \epsilon > 0$, $\exists\ \delta > 0$, such that $f^{-1}(B[f(a); \epsilon]) \supset B[a; \delta]$". It satisfies the condition (b).

Also, by theorem 5.2D. "If f is continious at a iff $\lim_{n\to\infty} x_n = a \Rightarrow \lim_{n\to\infty} f(x_n) = f(a)$", so, it satisfies the condition (c).

Hence f is cts at $a \in M$, Hence the proof of the theorem.

5.3.D. Theorem:

Let $\langle M_1, \rho_1 \rangle$, $\langle M_2, \rho_2 \rangle$ be metric spaces and let $f: M_1 \to M_2$, $g: M_2 \to M_3$. If f is cts at $a \in M_1$ and g is cts at $f(a) \in M_2$, then $g \circ f$ is cts at a.

Proof:

By (C) of 5.3 C all we need show is that

$$\lim_{n\to\infty} g[f(x_n)] = g[f(a)]$$

whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in M_1 such that

$$\lim_{n\to\infty} x_n = a.$$

The proof then proceeds exactly like the proof of 5.2 E.

For real-valued function on metric spaces there is a generalization of 5.1 B. The following theorm may be easily deduced from 4.2 B.

5.3.E Theorem:

Let M be a metric space, and let f and g be real-valued function which are continuous at $a \in M$. Then $f+g$, $f-g$, and fg are also continuous at a. Futhermore, if $g(a) \neq 0$, then f/g is continuous at a.

$\rho_2[(f+g)(x), (f+g)(a)] < \epsilon$ $[\rho, f(x, a) < \delta]$

5.3. F. Defn:

Let M_1 and M_2 be metric spaces and let $f: M_1 \to M_2$. we say that f is a continous function from M_1 into M_2 (or, more simply, f is cts on M_1), if f is cts at each point in M_1