5.3.G Theorem :

If f and g are continous functions from a metric space M_1 into a metric space M_2, then so are $f+g$, $f-g$ and fg. Further more, if $g(x) \neq 0$ $(x \in M_1)$ if f/g is also cts at each point on in M_1.

Proof:

Any polynomial function f [that is; $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$] is thus a cts fn on R^1. For constant fns are cts on R^1 and so is the function $g(x) = x$. The fn f can be written as a sum of products of those kinds of function and is thus, by continous.

The function h defined by $h(x) = (1+x^2)(1+x^2)$ can be written f/g where f and g are polynomials. Since $g(x)$ is never zero, it follows that h is continous on R^1.

Here is a more curious illustration. Let f be any function from the metric space R_1 into a metric space M. We have already observed that for any $a \in R_d$, the open ball $B[a;1]$ contains only the point a. Thus, for any $\epsilon > 0$, the inverse image under f of $B[f(a);\epsilon]$ certainly contains $B[a;1]$. By (b) of 5.3C, this shows that f is cts at a. Since a was an arbitary pt in R_d.

5.4. Open sets :

5.4. A. Defn :

* Let M be a metric space we say that the subset G of M is an open subset of M (or, more simply, that G is open) if for every $x \in G$ there exists a number $r > 0$ such that the entire open ball $B[x; r]$ is contained in G. $B \subset G$.

* consider R_1, If $a \in R_1$, then $\{a\} = B[a; 1]$ and hence $\{a\}$ is an open set in R_1. That is, any set with only one point in it is open in R.

5.4. B. Theorem :

In any metric space $\langle M, \rho \rangle$ both M and the empty set $\emptyset$ as open sets.

Proof :

If $x \in M$ then every open ball $B[x; r]$ is contained in M. Hence M is open. The empty set $\emptyset$ is open simply because there is no x in $\emptyset$ and hence every $x \in \emptyset$ satisfies the condition of 5.4. A.

5.4. C. Theorem :

Let $\mathcal{F}$ be any nonempty family of open subsets of a metric space M. Then $\bigcup_{\mathcal{F}} G$ is also an open subset of M.

Proof :

Let $H = \bigcup_{\mathcal{F}} G$. we may assume that at least one $G \in \mathcal{F}$ is an empty. choose any $x \in H$. we must then that there is an open ball $B[x; r]$ contained in H. But if $x \in H$, then, $x \in G$ for some $G \in \mathcal{F}$. Since G is open there is some $B[x; r]$ with $B[x; r] \subset G$. But $G \subset H$ and then $B[x; r] \subset H$, which is what we wished to show.

5.4. D. Theorem:

Every subset of R_d is open.

proof:

we have, that all one-point subsets of R_d are open.

But any subset G of R_d is obviously a union of such sets, then G is open.

It is not true, however, that the intersection of an infinite number of open sets in a metric space is always open. In R', for, ex. If I_n denotes the open interval $(-1/n, 1/n)$, then, $\bigcap_{n=1}^{\infty} I_n$ contains only 0 and is therefore not open.

5.4. E. Theorem:

If G_1 and G_2 are open subsets of the metric space M, then $G_1 \cap G_2$ is also open.

Proof:

we may assume that $G_1 \cap G_2 \neq \emptyset$. If $x \in G_1 \cap G_2$, we must find an open ball $B[x;r]$ contained in $G_1 \cap G_2$. Since $x \in G_1$ and G_1 is open, there is a ball $B[x;r_1]$ with $B[x;r_2] \subset G_2$. Similarly, there is a ball $B[x;r_2]$ with $B[x;r_2] \subset G_1$. Thus if $r = \min(r_1, r_2)$, then $B[x;r]$ is contained in G_1 and G_2 and thus $B[x;r] \subset G_1 \cap G_2$. This completes the proof.

5.4. F. Theorem:

Every open subset G of R' can be written $G = \bigcup I_n$ where $I_1, I_2, \ldots$ are a finite number or a countable number of open intervals which are mutually disjoint. (That is, $I_m \cap I_n = \emptyset$ if $m \neq n$).

Proof :

If $x \in G$, then there is an open interval (open ball) B containing x such that $B \subset G$. Let I_n denote the largest open interval containing x such that $I_n \subset G$. [I_n may be an unbounded interval]

Then $G = \bigcup_{z \in \theta} I_z$, Now if $x \in G$, $y \in G$, then either $I_z = I_y$ or $I_z \cap I_y = \emptyset$. For if $I_z \neq I_v$ and $I_z \cap I_v \neq \emptyset$.

Then $I_z \cup I_v$ would be an open interval contained in G which is larger than I_z.

This contradicts the defn of I_z. Finally, each I_z contains a rational number. Since disjoint intervals cannot contain the same rational and since there are only countably many rationals, there cannot be uncountably many mutually disjoint intervals I_z.

*We can use the notion of open set to give a necessary and sufficient condition that a function on a metric space be cts. The following theorem is fundamental.

5.4.G. Theorem :

Let $\langle M_1, \rho_1 \rangle$ and $\langle M_2, \rho_2 \rangle$ be metric spaces and let $f : M_1 \to M_2$. Then f is cts on M_1 if and only if $f^{-1}(G)$ is open in M_1, whenever G is open in M_2. (i.e. f is cts if and only if the inverse image of every open set is open).

Proof :

Suppose first that f is cts on M_1. We show that if G is open in M_2 then $f^{-1}(G)$ is open in M_1. Thus, if $x \in f^{-1}(G)$, we must find an open ball $B[x; r]$ contained in $f^{-1}(G)$.

Now, since $x \in f^{-1}(G)$ then $y = f(x) \in G$. Hence there is an open ball $B[y;s]$ contained in G ($\because G$ is open in M_2).

Then, $f^{-1}(B[y;s])$ contains some $B[x;r]$. ~~Hence $f^{-1}(G) \supset f^{-1}B(y;s)$~~

Hence, $f^{-1}(G) \supset f^{-1}(B[y;s]) \supset B[x;r]$ which is what we to show.

* Now, suppose $f^{-1}(G)$ is open in M_1 whenever G is open in M_2. To show that f is cts on M_1 it is sufficient to show that f is cts at an arbitary point $a \in M_1$.

Let $R = B[f(a); c]$ be any ball about $f(a)$. Then R is open in M_2 and so, by assumption, $f^{-1}(R)$ is open in M_1.

Since $a \in f^{-1}(R)$ and $f^{-1}(R)$ is open, there is an open ball $B[a;\delta]$ contained in $f^{-1}(R]$. But, then by (b) of 5.3.C, f is cts at a. This completes the proof.

5.5. Closed sets:

5.5.A. Defn:

Let E be a subset of the metric space M. A point $x \in E$ called a limit point of E if there is a sequence $\{x_n\}_{n=1}^{\infty}$ of points of E which converges to x. The set $\bar{E}$ of all limit points of E is called the closure of E.

5.5.C. Definition:

* Let E be a subset of the metric space M. Then E is closed subset of M if $E = \bar{E}$.

5.5.D. Theorem:

Let E be a subset of the metric space M. Then the point $x \in M$ is a limit point of E if and only if every open ball $B[x;r]$ about x contains at least one point of E.

proof:

Suppose x is a limit point of E. Then there is a sequence $\{x_n\}_{n=1}^{\infty}$ of points of E that converges to x. If $B[x;r]$ is any open ball about x, then $B[x;r]$ contains x_n for any n such that $\rho(x_n, x) < r$. Hence $B[x;r]$ contains a point of E.

Conversely, let $x \in M$ and suppose every $B[x;r]$ contains a point of E. Then for $n \in I$, the open ball $B[x;1/n]$ contains a point $x_n \in E$. The sequence $\{x_n\}_{n=1}^{\infty}$ obviously converges to x [since $\rho(x, x_n) < 1/n$], and hence x is a limit point of E. The proof is complete.

5.5. E. Theorem:

If E is any subset of a metric space M, then $\overline{E}$ is closed. That is, $\overline{\overline{E}} = \overline{E}$.

proof:

Since $\overline{E} \subset \overline{\overline{E}}$ we need only prove $\overline{E} \supset \overline{\overline{E}}$. Take any $x \in \overline{\overline{E}}$. To show that $x \in \overline{E}$ is enough to show that any open ball $B[x;r]$ contains a point of E. Since $x \in \overline{\overline{E}}$, the ball $B[x;r]$ contains a point $y \in \overline{E}$. Let $s = \rho(x,y)$ and choose any positive number t with $t < r-s$. Since $y \in \overline{E}$ the Ball $B[y;t]$ contains, a point $z \in E$. But $\rho(x,y) = s$, $\rho(y,z) < t < r-s$ and so,

$\rho(x,z) \le \rho(x,y) + \rho(y,z) < s + r - s = r$.

Hence $z \in B[x;r]$. Thus $B[x;r]$ contains a point of E, which ~~is what we~~ is the proof.

5.5. G. Theorem:

If F_1 and F_2 are closed subsets of the metric space M, then $F_1 \cup F_2$ is also closed.

proof:

Let $x \in \overline{F_1 \cup F_2}$. Then there is a sequence $\{x_n\}_{n=1}^{\infty}$ of points in $F_1 \cup F_2$ converging to x. But $\{x_n\}_{n=1}^{\infty}$ must have a subsequence consisting wholly of points in F_1 or a subsequence consisting of points in F_2. Since any subsequence of $\{x_n\}_{n=1}^{\infty}$ must converge to x, this shows that either $x \in \overline{F_1} = F_1$ or $x \in \overline{F_2} = F_2$. Thus $x \in F_1 \cup F_2$. Hence $F_1 \cup F_2 \supset \overline{F_1 \cup F_2}$. The proof is complete.

5.5 H. Theorem:

If $\mathcal{F}$ is any family of closed subsets of a metric space M, then $\bigcap_{F \in \mathcal{F}} F$ is also closed.

proof:

Let $x \in \overline{\bigcap_{F \in \mathcal{F}} F}$. Then any ball $B[x;r]$ contains a point $y \in \bigcap_{F \in \mathcal{F}} F$. Thus for any $F \in \mathcal{F}$, the ball $B[x;r]$ contains a point of F, namely y. Hence $x \in \overline{F} = F$. Thus x lies in every $F \in \mathcal{F}$ and so $x \in \bigcap_{F \in \mathcal{F}} F$. This proves $\bigcap_{F \in \mathcal{F}} F \supset \overline{\bigcap_{F \in \mathcal{F}} F}$ and thus $\bigcap_{F \in \mathcal{F}} F$ is closed.

5.5.I. Theorem :

Let G be an open subset of the metric space M. Then $G' = M - G$ is closed. Conversely, if F is a closed subset of M, then $F' = M - F$ is open.

Proof :

Suppose first that G is open. If $x \in G$, then there is a ball $B = B[x; r]$ which lies entirely in G. Hence B contains no point of G'. The point x cannot be a limit point of G'. Thus no point in G is a limit point of G', so G' contains all its limits points and is thus closed.

Now, suppose F is closed. If $y \in F'$ there must be a Ball $B[y; r]$ which contains no point of F. For otherwise y would be a limit point of F. We would then have $y \in F$, whic contradic $y \in F'$. Thus for every $y \in F'$ there is a Ball $B[y; r]$ lying entirely in F'. Hence F' is open.

5.5.J. Theorem :

Let $\langle M_1, \rho_1 \rangle$ and $\langle M_2, \rho_2 \rangle$ be metric spaces, and let $f : M_1 \to M_2$. Then f is cts on M_1 if and only if $f^{-1}(F)$ is a closed subset of M_1 whenever F is a closed subset of

Proof :

Suppose first that f is cts on M_1. If $F \subset M_2$ is a closed set, then F' is open. Then $f^{-1}(F)$ is open in M_1.

But since $F \cup F' = M_2$. we have, $f^{-1}(F) \cup F'(F') = f^{-1}(M$

That is, $F^{-1}(F) \cup f^{-1}(F') = M_1$. Hence $f^{-1}(F)$ is the complement of $f^{-1}(F')$. Since $f^{-1}(F')$ is open, then

$f^{-1}(F)$ is closed, which is what to wished to show.

5.5.k. Theorem:

Let f be a 1-1 function from a metric space M_1 onto a metric space M_2. Then if f has any one of the following properties, it has them all.

Proof:

(a) Both f and f^{-1} are cts (on M_1 and M_2, respectively).

(b) The set $G \subset M_1$ is open if ~~band~~ and only if its image $f(G) \subset M_2$ is open.

(c) The set $F \subset M_1$ is closed if and only if its image $f'(F)$ is closed.

5.5.L. Defn:

* If f has any one of the properties in 5.5 k we call f a homomorphism from M_1 onto M_2. If a homeomorphism from M_1 onto M_2 exists, we say that M_1 and M_2 are homeomorphic.

5.5.M. Defn.

* Let M be a metric space. The subset A of M is said to be ~~dose~~ dense in M if $\overline{A} = M$.