UNIT - 03

System of first order equation

Linear equation system

The only first order equation unknown function of the form.

$$\frac{d(x)}{dt} = f(t, x, y) \qquad \frac{dy}{dt} = G(t, x, y)$$

$$\frac{dx}{dt} = a_1 t(x) + b_1 t(y) + F_1(t) \qquad \frac{dy}{dt} = a_2 t(x) + b_2 t(y) + F_2(t)$$

where t is a independent variable x and y are dependent variable.

If $f_1(t)$ and $F_2(t)$ are identically zero Then the system is called homogeneous otherwise it is said to be a non-homogeneous.

Theorem - 1

If the homogeneous system

$$\frac{dx}{dt} = a_1(t)x + b_1(t)y + F_1(y)$$

$$\frac{dy}{dt} = a_2(t)x + b_2(t)y + F_2(y) \quad , x = x_1(t) \; ; y = y_1(t)$$

and $x = x_2(t)$; $y = y_2(t)$ on $[a, b]$ Then

$x = c_1 x_1(t) + C_2 x_2(t)$, $y = C_1 y_1(t) + C_2 y_2(t)$. is also a solution on $[a, b]$ for any Constant C_1 & C_2 are Constant.

Given $x = x_1(t)$, $y = y_1(t)$ is solution of ①

also $x = x_2(t)$, $y = y_2(t)$. is soln of ①

$$x_1' = a_1 x_1 + b_1 y_1 \rightarrow ① \qquad x_2' = a_1 x_2 + b_1 y_2 \rightarrow ②$$

$$y_1' = a_2 x_1 + b_2 y_1 \rightarrow ③, \qquad y_2' = a_2 x_2 + b_2 y_2 \rightarrow ④$$

Consider,

$$x = C_1 x_1 + C_2 x_2 \rightarrow ⑤ \qquad y = C_1 y_1 + C_2 y_2 \rightarrow ⑥$$

Differentiate with respect to x & y we get

$$x' = C_1 x_1' + C_2 x_2' \qquad y' = C_1 y_1' + C_2 y_2'.$$

using ① & ③ we get

$x' = c_1(a_1x_1 + b_1y_1) + c_2(a_1x_2 + b_1y_2)$

$= c_1a_1x_1 + c_1b_1y_1 + c_2a_1x_2 + c_2b_1y_2$

$= a_1(c_1x_1 + c_2x_2) + b_1(c_1y_1 + c_2y_2)$

from ⑤ & ⑥ we get

$x' = a_1x + b_1y$

Now,

$y' = c_1y_1' + c_2y_2'$

using ② & ④ we get

$y' = c_1(a_2x_1 + b_2y_1) + c_2(a_2x_2 + b_2y_2)$

$= c_1a_2x_1 + c_1b_2y_1 + c_2a_2x_2 + c_2b_2y_2$

$= a_2(c_1x_1 + c_2x_2) + b_2(c_1y_1 + c_2y_2)$

from ⑤ & ⑥ we get

$y' = a_2x + b_2y.$

Hence,

$x = c_1x_1(t) + c_2x_2(t)$

$y = c_1y_1(t) + c_2y_2(t).$

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problems.

1) S.T $x = e^{2t}$, $x = e^{3t}$ & $y = 2e^{2t}$, $y = e^{3t}$. are the soln of homogeneous equation $\frac{dx}{dt} = 4x - y$, $\frac{dy}{dt} = 2x + y$.

Given $\frac{dx}{dt} = 4x - y \rightarrow ①$ $\quad \frac{dy}{dt} = 2x + y \rightarrow ②$

$Dx = 4x - y$ $\quad Dy = 2x + y$

$Dx - 4x + y = 0$ $\quad Dy - 2x - y = 0$

$x(D-4) + y = 0 \rightarrow ③$ $\quad y(D-1) - 2x = 0 \rightarrow ④$

$③ \times (D-1) \Rightarrow x(D-1)(D-4) + \cancel{y(D-1)} = 0$

$④ \Rightarrow \quad \underset{(+)}{-2x} \quad \underset{(-)}{+} \cancel{y(D-1)} = 0$

$x(D-1)(D-4) + 2x = 0$

$$(D^2 - 4D - D + 4)x + 2x = 0$$
$$x(D^2 - 5D + 4 + 2) = 0$$
$$x(D^2 - 5D + 6) = 0$$
$$x = 0, \quad (D-3)(D-2) = 0$$

$6 < {-3 \atop -2}$, -5

Hence $\boxed{D = 2, 3}$

$\underline{m_1 = 3} \qquad \underline{m_2 = 2}$

The roots are real.

$$x = Ae^{m_1 x} + Be^{m_2 x}$$
$$x = Ae^{3t} + Be^{2t}$$

Now,

$2 \times (3)$ $\qquad 2x(\cancel{D-4}) + 2y = 0$

$(D-4) \times (4)$ $\qquad -2x(\cancel{D-4}) + y(D-1)(D-4) = 0$

$$y(D-1)(D-4) + 2y = 0$$
$$y(D^2 - D + 4 - 4D) + 2y = 0$$
$$y(D^2 - 5D + 4 + 2) = 0$$
$$y(D^2 - 5D + 6) = 0$$
$$y = 0, \quad D = 3, 2$$
$$m_3 = 3, \quad m_4 = 2$$

The roots are real.

$$y = Ce^{3t} + De^{2t}$$

Now,

$$\frac{dx}{dt} = 4x - y$$

$$\frac{d}{dt}(Ae^{3t} + Be^{2t}) = 4(Ae^{3t} + Be^{2t}) - (Ce^{3t} + De^{2t})$$
$$3Ae^{3t} + 2Be^{2t} = 4Ae^{3t} + 4Be^{2t} - Ce^{3t} - De^{2t}$$
$$3Ae^{3t} + 2Be^{2t} - 4Ae^{3t} - 4Be^{2t} = -Ce^{3t} - De^{2t}$$
$$-Ae^{3t} - 2Be^{2t} + Ce^{3t} + De^{2t} = 0$$
$$e^{3t}(C - A) + e^{2t}(D - 2B) = 0$$

If $\boxed{A = 1}$ then $\boxed{C = 1}$

If $\boxed{B = 1}$ then $\boxed{D = 2}$

Hence $x = e^{3t} + e^{2t}$

$y = e^{3t} + 2e^{2t}$.

2. S.T $x = 2e^{4t}$, $x = e^{-t}$, $y = 3e^{4t}$ $y = -e^{-t}$ are soln of homogeneous system of $\frac{dx}{dt} = x+2y$, $\frac{dy}{dt} = 3x+2y$

Given

$\frac{dx}{dt} = x+2y \rightarrow (1)$ $\frac{dy}{dt} = 3x+2y \rightarrow (2)$

let $D = d/dt$

$Dx = x+2y$ $Dy = 3x+2y$

$Dx - x - 2y = 0$ $Dy - 2y - 3x = 0$

$x(D-1) - 2y = 0 \rightarrow (3)$ $y(D-2) - 3x = 0 \rightarrow (4)$

$(3) \times 3$ $3x(D-1) - 6y = 0$

$(4) \times (D-1)$ $-3x(D-1) + y(D-2)(D-1) = 0$

$y(D-2)(D-1) - 6y = 0$

$y(D^2 - 2D + 2 - D) - 6y = 0$

$y(D^2 - 3D + 2 - 6) = 0$

$y(D^2 - 3D - 4) = 0$

-4 $+1$ -4 -3

$\boxed{y=0}$ $D = -1, +4$

Hence $y = Ae^{4t} + Be^{-t}$

Now,

$(3) \times D-2$ $x(D-1)(D-2) - 2y(D-2) = 0$

$(4) \times 2$ $-6x + 2y(D-2) = 0$

$x(D-1)(D-2) - 6x = 0$

$x(D^2 - D + 2 - 2D) - 6x = 0$

$x(D^2 - 3D - 4) = 0$

$\boxed{x=0}$ $\boxed{D = -1, 4}$

$x = Ce^{4t} + De^{-t}$

Now,

$$\frac{dx}{dt} = x + 2y$$

$$\frac{d}{dt}\{ce^{4t}+De^{-t}) = ce^{4t}+De^{-t}+2Ae^{4t}+2Be^{-t}$$

$$4ce^{4t}-De^{-t} = ce^{4t}+De^{-t}+2Ae^{4t}+2Be^{-t}$$

$$4ce^{4t}-De^{-t}-ce^{4t}-De^{-t} = 2Ae^{4t}+2Be^{-t}$$

$$3ce^{4t}-2De^{-t}-2Ae^{4t}-2Be^{-t}=0$$

$$e^{4t}(3c-2A)+e^{-t}(-2D-2B)=0$$

Hence $3C = 2A$, $-2D = 2B$

if $A=3$ then $C=2$

If $D=1$ then $B=-1$

Hence,

$$x = Ce^{4t}+De^{-t} \quad , \quad y = Ae^{4t}+Be^{-t}$$

$$x = 2e^{4t}+e^{-t} \quad , \quad y = 3e^{4t}-e^{-t}$$

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3. S.T $x = e^{4t}$, $y = -e^{-2t}$ are soln of the homogeneous system $\frac{dx}{dt} = x+3y$ & $\frac{dy}{dt} = 3x+y$ find particular soln of $x = x(t)$, $y = y(t)$ of this system for which $x(0)=5$, $y(0)=1$.

Given, $\frac{dx}{dt} = x+3y \rightarrow (1)$; $\frac{dy}{dt} = 3x+y \rightarrow (2)$

Let $d/dt = D$ then

$$D(x) = x+3y \qquad\qquad Dy = 3x+y$$

$$Dx-x-3y = 0 \qquad\qquad Dy-y-3x=0$$

$$x(D-1)-3y = 0 \longrightarrow (3) \qquad\qquad y(D-1)-3x=0 \longrightarrow (4)$$

(3) $\times 3$ $\quad 3x(D-1)-9y=0$

(4) $\times (D-1)$ $\quad -3x(D-1)+y(D-1)(D-1)=0$

$$y(D-1)^2-9y=0$$

$$y(D^2-2D+1-9)=0$$

$$y(D^2-2D-8)=0$$

$y=0 \quad (D-4)(D+2)=0$

$$\boxed{D = 4, -2}$$

$$y = Ae^{4t} + Be^{-2t}$$

(3) $\times (D-1) \rightarrow \quad x(D-1)(D-1) - 3\cancel{y(D-1)} = 0$

(4) $\times 3 \rightarrow \quad -9x + \cancel{3y(D-1)} = 0$

$$x(D-1)^2 - 9x = 0$$

$$x(D^2 - 2D - 1 - 9) = 0$$

$x=0 \quad (D-4)(D+2)=0$

$$\boxed{D = 4, -2}$$

Hence

$$x = Ce^{4t} + De^{-2t}$$

Now,

$$\frac{dx}{dt} = x + 3y$$

$$\frac{d}{dt}(Ce^{4t} + De^{-2t}) = Ce^{4t} + De^{-2t} + 3(Ae^{4t} + Be^{-2t})$$

$$4Ce^{4t} - 2De^{-2t} = Ce^{4t} + De^{-2t} + 3Ae^{4t} + 3Be^{-2t}$$

$$4Ce^{4t} - 2De^{-2t} - Ce^{4t} - De^{-2t} - 3Ae^{4t} - 3Be^{-2t} = 0$$

$$3Ce^{4t} - 3De^{-2t} - 3Ae^{4t} - 3Be^{-2t} = 0$$

$$e^{4t}(3C - 3A) + e^{-2t}(-3D - 3B) = 0$$

$$3e^{4t}(C - A) + 3e^{-2t}(-D - B)$$

$\boxed{C = A}$ $\quad \boxed{B = -D}$

if $C = 1$ then $A = 1$

if $B = 1$ then $D = -1$

Hence $\quad x = e^{4t} - e^{-2t}$

$$y = e^{4t} + e^{-2t}$$

4. Solve $\frac{dx}{dt} = x+y$, $\frac{dy}{dt} = y$

$\frac{dx}{dt} = x+y \rightarrow (1)$ $\qquad$ $\frac{dy}{dt} = y \rightarrow (2)$

let $d/dt = D$

$Dx = x+y$ $\qquad$ $Dy = y$

$Dx - x - y = 0$ $\qquad$ $Dy - y = 0$

$x(D-1) - y = 0 \rightarrow (3)$ $\qquad$ $y(D-1) = 0 \rightarrow (4)$

$(3) \times D-1 \rightarrow \quad x(D-1)^2 - y(D-1) = 0$

$(4) \rightarrow \quad y(D-1) = 0$

$$x(D-1)^2 = 0$$

$\boxed{x=0}$ $\boxed{D = 1, 1.}$

$$\boxed{x = Ae^t + Be^t}$$

from (4)

$y(D-1) = 0$

$\boxed{y=0}$ $\boxed{D=1}$

Hence $\boxed{y = ce^t}$

5. Find the general solution of system

$\frac{dx}{dt} = x$ $\qquad$ $\frac{dy}{dt} = y$

Given $\frac{dx}{dt} = x \rightarrow (1)$ $\qquad$ $\frac{dy}{dt} = y \rightarrow (2)$

$Dx = x$ $\qquad$ $Dy = y$

$Dx - x = 0$ $\qquad$ $Dy - y = 0$

$x(D-1) = 0$ $\qquad$ $y(D-1) = 0$

$D = 1$ $\qquad$ $D = 1$

$x = Ae^t$ // $\qquad$ $y = Be^t$ //

Linear system of Ist order equation homogeneous system with Constant Co-efficients

we are how the position to give a Complete explict soln of simple System $\frac{dx}{dt} = a_1 x + b_1 y \rightarrow (1)$ $\frac{dy}{dt} = a_2 x + b_2 y$

wher a_1, a_2, b_1 and b_2 are given Constants soln of (1) having the form

$$x = Ae^{mt} , \quad y = Be^{mt} \longrightarrow (2)$$

Differentiate w.r.t t :,

$$\frac{dx}{dt} = mAe^{mt} , \quad \frac{dy}{dt} = mBe^{mt} \longrightarrow (3)$$

Substitute (1) we get

$$mAe^{mt} = a_1 Ae^{mt} + b_1 Be^{mt} \longrightarrow (4)$$

$$mBe^{mt} = a_2 Ae^{mt} + b_2 Be^{mt} \longrightarrow (5)$$

Divided by e^{mt} on both side we get linear algebric system

$$Am = a_1 A + b_1 B \longrightarrow (6)$$

$$Bm = a_2 A + b_2 B \longrightarrow (7)$$

from (6) & (7) we get

$$Am - a_1 A = b_1 B \quad ; \quad Bm - a_2 A = b_2 B$$

$$-Am + a_1 A + b_1 B = 0 \qquad -Bm + a_2 A + b_2 B = 0$$

$$A(a_1 - m) + b_1 B = 0 \rightarrow (8) \quad B(b_2 - m) + a_2 A = 0 \rightarrow (9)$$

In the unknown A and B . It is Clear that (3) has trivial soln $A = B = 0$, which makes (2) the trivial soln of (1) whenever the determinant of Coefficient vanishes.

$$\begin{vmatrix} a_1 - m & b_1 \\ a_2 & b_2 - m \end{vmatrix} = 0$$

When this determinants is expand we get the quadratic equation.

$$(a_1 - m)(b_2 - m) - a_2 b_1 = 0$$

$$a_1 b_2 - m b_2 + m^2 - a_1 m - a_2 b_1 = 0$$

$$m^2 - (a_1 + b_2)m + (a_1 b_2 - a_2 b_1) = 0$$

for the unknown m we call this the auxillary equation of system ①.

Let m_1, m_2 be the root of the equation ① if we replace m in ⑧ & ⑨ by m. Then we know that resulting equation have a non trivial solution A_1, B_1 so that

$$x = A_1 e^{mt} \qquad y = B_1 e^{mt}$$

is a non trivial solution of ①.

By proceeding similarly with m_2 we find another non trivial soln.

$$x = A_2 e^{m_2 t} \qquad y = B_2 e^{m_2 t}$$

In order to make such that we Obtain two L.I soln, Hence the general solution, it is neccessarily to examine in detail each of there possibilities for m_1 and m_2.

Distinct real root

Case (i)

When m_1 and m_2 roots are distinct, so the soln are

$$x_1 = A_1 e^{mt}, \quad x_2 = A_2 e^{mt} \longrightarrow ①$$

$$y_1 = B_1 e^{mt}, \quad y_2 = B_2 e^{mt} \longrightarrow ②$$

Hence the general soln of ① is

$$x = C_1 A_1 e^{m_1 t} + C_2 A_2 e^{m_2 t}$$
$$y = C_1 B_1 e^{m_1 t} + C_2 B_2 e^{m_2 t}$$

Case (ii) roots are equal

When the roots are real and equal then

$$x_1 = A_1 e^{mt} \qquad y_1 = B_1 e^{mt}$$

The second solution will be of the

$$x_2 = A_2 e^{mt} \qquad y_2 = B_2 e^{mt}$$

unfortunately the matter we must actually look for the second soln of the form.

$$x = (A_1 + A_2) e^{mt}$$
$$y = (B_1 + B_2) e^{mt}$$

Hence the general soln is

$$x = C_1 A_1 e^{mt} + C_2 A_2 e^{mt}$$
$$y = C_1 B_1 e^{mt} + C_2 B_2 e^{mt}$$

Case (iii) Roots are distinct

When m_1, m_2 are distinct Complex numbers then they can be written in form $a \pm ib$

Here a & b are real numbers and $b \neq 0$.

The two linear independent soln

$$x = A_1^* = A_1 + iA_2 \qquad B_1^* = B_1 + iB_2$$
$$A_2^* = A_3 + iA_4 \qquad B_2^* = B_3 + iB_4$$

note that soln of can be written as $x = (A_1 + iA_2) e^{(a+ib)t}$