Hence the general soln of ① is

$x = C_1 A_1 e^{m_1 t} + C_2 A_2 e^{m_2 t}$

$y = C_1 B_1 e^{m_1 t} + C_2 B_2 e^{m_2 t}$

Case (ii) roots are equal

When the roots are real and equal then

$x_1 = A_1 e^{mt}$ $\quad y_1 = B_1 e^{mt}$

The second solution will be of the

$x_2 = A_2 e^{mt}$ $\quad y_2 = B_2 e^{mt}$

unfortunately the matter we must actually look for the second soln of the form.

$x = (A_1 + A_2) e^{mt}$

$y = (B_1 + B_2) e^{mt}$

Hence the general soln is

$x = C_1 A_1 e^{mt} + C_2 A_2 e^{mt}$

$y = C_1 B_1 e^{mt} + C_2 B_2 e^{mt}$

Case (iii) Roots are distinct

When m_1, m_2 are distinct complex numbers then they can be written in form $a \pm ib$

Here a & b are real numbers and $b \neq 0$.

The two linear independent soln

$x = A_1^* = A_1 + iA_2 \quad B_1^* = B_1 + iB_2$

$A_2^* = A_3 + iA_4 \quad B_2^* = B_3 + iB_4$

note that soln of can be written as $x = (A_1 + iA_2) e^{(a+ib)t}$

$y = (B_1 + iB_2) e^{(a+ib)t}$

then,

$x = (A_1 + iA_2) e^{at} \cdot e^{ibt}$

$y = (B_1 + iB_2) e^{at} \cdot e^{ibt}$

$x = e^{at} (A_1 + iA_2) \cdot (\cos bt + i \sin bt)$

$x = e^{at} \left(A_1 \cos bt + A_1 i \sin bt + i A_2 \cos bt - A_2 \sin bt \right)$

$x = e^{at} \left[C_1 \left(A_1 \cos bt - A_2 \sin bt \right) + i C_2 \left(A_1 \sin bt + A_2 \cos bt \right) \right]$

~~Two real values~~

$y = e^{at} \left[\left(B_1 \cos bt - B_2 \sin bt \right) + i C_2 \left(B_1 \sin bt + B_2 \cos bt \right) \right]$

To real values are

$x = e^{at} (A_1 \cos bt - A_2 \sin bt)$

$y = e^{at} (B_1 \cos bt - B_2 \sin bt)$

It can be show that soln ar L.I

therefore the general soln of ① is $x = C_1 e^{at} \left[(A_1 \cos bt - A_2 \sin bt) + i C_2 (A_1 \sin bt + A_2 \cos bt) \right]$,

Problems

1. $\frac{dx}{dt} = 7x + 6y$, $\frac{dy}{dt} = 2x + 6y$ find general soln.

Given $\frac{dx}{dt} = 7x + 6y \rightarrow ①$, $\frac{dy}{dt} = 2x + 6y \rightarrow ②$

Here $a_1 = 7$, $a_2 = 2$, $b_1 = 6$, $b_2 = 6$.

The linear algebraic equation is

$(a_1 - m) A + b_1 B = 0$

$a_2 A + (b_2 - m) B = 0$

$(7-m)A + 6B = 0 \longrightarrow (3)$

$2A + (6-m)B = 0 \longrightarrow (4)$

The auxilary equation is

$m^2 - (a_1+b_2)m + (a_1b_2 - a_2b_1) = 0$

$m^2 - (7+6)m + [(7\times6) - (2\times6)] = 0$

$m^2 - 13m + (42-12) = 0$

$m^2 - 13m + 30 = 0$

Hence $\boxed{m = 10, 3}$

Case (i)

Let $m = 10$

from (3) $-3A + 6B = 0$

$-3A = -6B$

$A = 2B$

Hence $\boxed{A=2}$ $\boxed{B=1}$

Hence First L. I soln is

$x = A_1 e^{m_1 t}$, $y = B_1 e^{m_1 t}$

$x = 2e^{10t}$ $y = e^{10t}$ //

Case (ii)

Let $m_2 = 3$, we get

$4A + 6B = 0$

$4A = -6B$

$2A = -3B$

if $A_2 = 3$ then $B_2 = -2$

Substitute the value of $n=3$, $B=2$ $m=3$ in below equation

$x = A_2 e^{m_2 t}$ $y = B_2 e^{m_2 t} \longrightarrow (5)$

$x = 3e^{3t}$ $y = -2e^{3t} \longrightarrow (6)$

equation (5) & (6) are linearly independent

Hence the general soln

$x = 2C_1 e^{10t} + 3C_2 e^{3t}$

$y = C_1 e^{10t} - 2C_2 e^{3t}$.

2. Find general soln of $\frac{dx}{dt} = x - 2y$

$\frac{dy}{dt} = 4x + 5y$.

$\frac{dx}{dt} = x - 2y$ $\frac{dy}{dt} = 4x + 5y$

Here $a_1 = 1$, $a_2 = 4$, $b_1 = -2$, $b_2 = 5$

The linear algebric system is given by $(a_1 - m)A + b_1 B = 0$

$a_2 A + (b_2 - m)B = 0$

$(1-m)A - 2B = 0$

$4A + (5-m)B = 0$

The auxillary equation is

$m^2 - (a_1+b_2)m + (a_1b_2 - a_2b_1) = 0$

$m^2 - 6m + 13 = 0$

$m = \frac{6 \pm \sqrt{3(6) - 4(13)}}{2}$

Hence $m = 3+2i$

$(1 - 3 - 2i)A - 2B = 0$

$(-2 - 2i)A - 2B = 0$

$-2(1+i)A = 2B$

if $A = 1$, $B = -(1+i)$

The solution are $x = e^{(3+2i)t}$

$y = -(1+i)e^{(3+2i)t}$

$x = e^{(3+2i)t}$

$= e^{3t} \cdot e^{2it}$

$= e^{3t}(\cos 2t + i\sin 2t)$

$y = -(1+i)e^{(3+2i)t}$

$= -(1+i)\, e^{3t} \cdot e^{2it}$

$= -(1+i)\, e^{3t}(\cos 2t + i \sin 2t)$

$= -e^{3t}(\cos 2t + i\sin 2t + i\cos 2t - \sin 2t)$

$= -e^{3t}\left[(\cos 2t - \sin 2t) + i(\sin 2t + \cos 2t)\right]$

equating real & imaginary part

$x_1 = e^{3t}\cos 2t \qquad x_2 = e^{3t}\sin 2t$

$y_1 = -e^{3t}(\cos 2t - \sin 2t) \quad y_2 = -e^{3t}(\cos 2t + \sin 2t)$

The general soln is

$x = e^{3t}(C_1 \cos 2t + C_2 \sin 2t)$

$y = e^{3t}\, C_1(\cos 2t - \sin 2t) + C_2(\cos 2t + \sin 2t)$.

CHAPTER-13

The method of successive approximation

Consider, the initial valuable problem of theorem.

$$\frac{dy}{dx} = f(x,y),\ y(x_0) = y_0 \longrightarrow ①$$

By integrating over the integral (x_0, x_1)

$$\int_{x_0}^{x_1} \frac{dy}{dx}\, dx = \int_{x_0}^{x} f(x,y)\, dx$$

$$[y]_{x_0}^{x_1} = \int_{x_0}^{x} f(x,y)\, dx$$

$$y(x_1) - y(x_0) = \int_{x_0}^{x} [f(x,y)]\, dx$$

$$y(x_1) = y(x_0) + \int_{x_0}^{x} f(x,y)\, dx$$

$$y(x_n) = y(x_0) + \int_{x_0}^{x} f(x,y)\, dx \longrightarrow ②$$

This solving of initial value problem of y to preterms of 'x' is absent the integral on R.H.S of (2) cannot be equivalent

Hence, exact value of 'y' can be obtained by determine the sequence of approximate soln and 2 as follows

As write approximately we put $y = y_0$ in interval on R.HS of ②

obtained

$$y_1(x) = y_0 + \int_{x_0}^{x} f(x_0, y_0)\, dx$$

Proceeding in this way the n^{th} approximation given by

$$y_n(x) = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx$$

The procedure is called picards method of successive approximations.

1. Find the exact soln of initial values problem $y' = y$, $y(0) = 1$.

Given $y(0) = 1$

Apply picards method

To calculate $y_1(x)$, $y_2(x)$ and $y_3(x)$ & compare the result there with exact soln

$$y' = y \longrightarrow (1)$$

$$\frac{dy}{dx} = y \Rightarrow \frac{dy}{y} = dx$$

integrating on both sides

$$\int \frac{dy}{y} = \int dx$$

$$\log y = x + \log c$$

$$\log y - \log c = x$$

$$\log \frac{y}{c} = x$$

$$y/c = e^x$$

$$\boxed{y = ce^x} \longrightarrow (2)$$

Since $x_0 = 0$, $y_0 = 1$

$$1 = ce^0$$

Hence $\boxed{C = 1}$

$$y = e^x$$

The equivatent interval soln is

$$y_n(x) = y(x_0) + \int_{x_0}^{x} f(x, y_{n-1})\, dx$$

put $\boxed{n=1}$ we get

$$y_1(x) = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$= 1 + \int_0^x 1\, dx$$

$$\boxed{y_1(x) = 1 + x}$$

put $\boxed{n=2}$ we get

$$y_2(x) = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

$$= 1 + \int_0^x (1+x)\, dx \quad \text{wheres } y_1(x) = 1+x$$

$$= 1 + \left[x + \frac{x^2}{2}\right]_0^x$$

$$\boxed{y_2(x) = 1 + x + \frac{x^2}{2}}$$

put $\boxed{n=3}$ we get

$$y_3(x) = y_0(x) + \int_0^x f(x, y_2)\, dx$$

$$= y_0 + \int_0^x \left(1 + x + \frac{x^2}{2}\right) dx$$

$$\boxed{y_3(x) = \left[1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right]}$$

Hence

$$y_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$y_n(x) = e^x$$

Hence $y_n(x) = y(x)$.

2. $y' = y^2$, $y(0) = 1$

Given $y' = y^2$, $y(0) = 1$

To calculate $y_1(x)$, $y_2(x)$, $y_3(x)$

Compare the result with exact

apply picards method.

$$\frac{dy}{dx} = y^2 \longrightarrow (1)$$

Integrating we get

$$\frac{dy}{y^2} = dx$$

$$\int \frac{dy}{y^2} = \int dx$$

$$-\frac{1}{y} = x + c \longrightarrow (2)$$

Since $y(0) = 1$.

Here $x_0 = 0$, $y_0 = 1$

$$-\frac{1}{y_0} = x_0 + c$$

$$\boxed{C = -1}$$

Put c value in (2) we get

$$-\frac{1}{y} = x + (-1)$$

$$\frac{1}{y} = 1 - x$$

$$y = \frac{1}{1-x} \longrightarrow (3)$$

The equivalent equation is

$$y_n(x) = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx$$

Put $\boxed{n=1}$ we get

$$y_1(x) = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$y_1(x) = 1 + \int_0^x dx$$

$$\boxed{y_1(x) = 1 + x}$$

Put $\boxed{n=2}$ we get

$$y_2(x) = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

$$= 1 + \int_0^x (1+x)^2\, dx$$

$$= 1 + \int_0^x (1 + x^2 + 2x)\, dx$$

$$= 1 + \left[x + \frac{x^3}{3} + \frac{\cancel{2}x^2}{\cancel{2}}\right]_0^x$$

$$\boxed{y_2(x) = 1 + x + x^2 + \frac{x^3}{3}}$$

put $\boxed{n=3}$ we get

$$y_3(x) = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx$$

$$= 1 + \int_0^x \left[1 + x + x^2 + \frac{x^3}{3}\right]^2 dx$$

$$= 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{3}\right)\left(1 + x + x^2 + \frac{x^3}{3}\right) dx$$

$$\boxed{y_3(x) = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots}$$

$$y_n(x) = 1 + x + x^2 + x^3 + \cdots$$

$$y_n(x) = (1-x)^{-1}$$

$$y_n(x) = \frac{1}{1-x}$$

Hence,

$$y_n(x) = y(x)$$ //

PICARDS THEOREM.

Statement

Let $f(x,y)$ on $\frac{\partial F}{\partial y}$ be continuous function of x,y on a closed rectangle R with sides parallel to the axis. if (x_0, y_0) is any interior point of R.

Then, there exists a number $h>0$ with the property that initial value problem $\frac{dy}{dx} = y' = f(x,y)$; $y(x_0) = y_0$

Has one and only soln $y = y(x)$ on the interval $|x-x_0| \leq h$.

proof:

$$y' = f(x,y) \quad ; \quad y(x_0) = y_0 \longrightarrow (1)$$

we know that,

every soln of (1) is also a continuous of equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt \longrightarrow (2)$$

and Conversly,

we conclude that (1) as a unique soln on an interval $|x-x_0| < h$ on the some interval if and only if equation (2) has a unique Continuous solution on same interval.

The sequence of function $y_n(x)$ is defined by $y_0(x) = y_0$

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0(t))\, dt$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\, dt$$

$$y_n(x) = y_0 + \int_{x_0}^{x} f[t, y_{n-1}(t)]\, dt \longrightarrow (3)$$

Convergence to a soln of eqn (2)

we next observe that $y_n(x)$ is the n^{th} partial sum of the series of function

Adding we get,

$$y_0(x) + \sum_{n=1}^{\infty} [y_n(x) - y_{n-1}(x)] = y_0(x) + [y_1(x) - y_0(x)] + \cdots [y_n(x) - y_{n-1}(x)] \longrightarrow (4)$$

So the Convergence of sequence (3) is equivalent this Convergence of their Series. Hence Complete the proof.

we produce a number $h>0$, that defines that interval $|x-x_0| < h$ and then we Show that on the interval the following statement are true.

(i) The series (4) converges to a function $y(x)$

(ii) $y(x)$ is a Continuous soln of (2)

(iii) $y(x)$ is the only continuous soln of (2)

The proof is hypothesis of the theorem.

Proof:

we are used to produce the number 'h'. we assumed that $f(x,y)$ and $\frac{\partial F}{\partial y}$ are Continuous on rectangle R. But R is closed and bounded so, each of these function is necessarily bounded on R.

If a constant μ an k

$$|f(x,y)| \leq \mu \longrightarrow (5)$$

$$\left|\frac{\partial}{\partial y} f(x,y)\right| \leq k \longrightarrow (6)$$

If (x, y_1) & (x, y_2) are distinct point in R with the same x-coordinate

Then, the mean value

$$|f(x,y_1) - f(x,y_2)| = \left|\frac{\partial}{\partial y} f(x,y)\right| |y_1 - y_2| \longrightarrow (7)$$

For some number y^* between y_1 and y_2 it is clear that from (6) & (7)

$$|f(x,y_1) - f(x,y_2)| \leq k |y_1 - y_2| \longrightarrow (8)$$

for any point (x, y_1), (x, y_2) in R that on some vertical line.

we now choose h be any possitive number.

$$kh < 1 \longrightarrow (9)$$

and the rectangle R defined by the inequality $|x - x_0| \leq h$

$|y - y_0| \leq \mu$

Take mod on both sides

Since (x_0, y_0) is an interior point of R.

(i) The series (4) converges to a function $(y(x))$ $f(x,y)$.

$$|y_0(x)| + |y_1(x) - y_0(x)| + |y_2(x) - y_1(x)| + \cdots$$
$$\cdots |y_n(x) - y_{n-1}(x)| \longrightarrow (10)$$

Convergent $y(x)$ has a graph line in order, Hence in R.

This is obvious $y_0(x) = y_0$

So that the point $[t, y_0(t)] \in R$

(5) $\Rightarrow$ $|f(t, y_0(t))| < \mu$

$$|y_1(x) - y_0(x)| \leq \left|\int_{x_0}^{x} f(t, y_0(t))\, dt\right|$$
$$\leq \mu h.$$

which proves the statement.

(ii) $y(x)$ is continuous soln for (2)

$(t, y(t))$ are in R

so, $|f(t, y(t))| \leq \mu$

$$|y_2(x) - y_0| = n \left|\int_{x_0}^{x} f(t, y_0(t))\right| dt$$
$$\leq \mu h$$

|||ly

$$|y_3(x) - y_0| = \left|\int_{x_0}^{x} f(t, y_2(t))\right| dt$$
$$\leq \mu h$$

Since Continuous function is closed on interval ~~has~~ has maximum μ and $y_1(x)$ is Continuous.

$$a = \max |y_1(x) - y_0|$$

write $|y_1(x) - y_0(x)| \leq a$ next the points $[t, y(t)]$ and $[t, y_0(t)]$ lie in R.

$⑧ \Rightarrow \left| f(t, y_1(t)) - f(t, y_0(t)) \right| \le ka \left| y_1(t) - y_0(t) \right|$

we have

$$\left| y_2(x) - y_1(x) \right| = \int_{x_0}^{x} f(t, y_1(t)) - f(t, y_0(t))\, dt$$

$|||^{ly}$ $\left| y_2(x) - y_1(x) \right| \le kan \le a(hk)$

$$\left| f(t, y_2(t)) - f(t, y_1(t)) \right| \le k \left| y_2(t) - y_1(t) \right|$$

$$\le k(kah)$$

By Continuing in this manner

$\left| y_n(x) - y_{n-1}(x) \right| \le a(kh)^{n-1}$ for every $n = 1, 2, \ldots$

each term of the series by Constant

$$|y_0(x)| + |y_1(x) - y_0(x)| + |y_2(x) - y_1(x)| \cdots + |y_n(x) - y_{n-1}(x)|$$

$$= \le (y_0 + a + akh) + 6\, a(kh)^2 \cdots a(kh)^{n-1}.$$

so eqn ⑩ converges by Comparison test equation.

① Convergence to sum we denote by $y(x)$ and $y_n(x) \to y(x)$

Sufficiently large if $\epsilon > 0$

$\exists$ +ve integer number no $\exists$ $g(n) \ge n$

$$|y(x) - y_n(x)| \le \epsilon \;\forall x$$

Since each $y_n(x)$ is clearly Continuous uniform of converges limit function $y(x)$ also Continuous it remaining to be P.T $y(x)$ actually a soln of ③:

$$y(x) - y_0 - \int_{x_0}^{x} f(t, y(t))\, dt = 0 \quad \longrightarrow ⑪$$

But W.K.T

$$y_n(x) - y_0 - \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt = 0$$

⑪ − ⑫ we get

$$y(x) - y_0 = \int_{x_0}^{x} f|(t, y(t))|\, dt$$

$$= \left| y(x) - y_n(x) + \int_{x_0}^{x} f(t, y_{n-1}(t)) \right|$$

$$\le |y(x) - y_0(x)| + \left| \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt \right|$$

Since $y(x)$ lies in R and hence in R. from ⑧

$$|f(x, y_1) - f(x, y_2)| \le k\, |y_1 - y_2|$$

We have,

$$\left| y(x) - y_0 - \int_{x_0}^{x} f(t, y(t))\, dt \right| \le |y(x) - y_n(x)| + kh\max|y_{n-1}(t) - y(x)| \longrightarrow ⑬$$

The uniformly Convergence of $y_n(x) + y(x)$ we can make RHS of ⑬ can be made smaller by taking large enough. The LHS of equation ⑬ must be equal to zero

$$y(x) - y_0 - \int_{x_0}^{x} f(t, y(t))\, dt = 0$$

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$$

proof is Complete.

(iii)

We assume that $y(x)$ is also Continuous soln of ② on interval $|x - x_0| \leq h$.

P.T $\bar{y}(x) = y(x)$ $\forall x$ in the interval

The graph $\bar{y}(x)$ lies in R.

Hence in R'.

We suppose that the graph of $y(\bar{x})$ leave R. since the func $\bar{y}(x)$ is Continuous of $y(x)=0$ $\exists$ a x.

$$\exists\ |x - x_0| \leq h \longrightarrow (a)$$

$$|\bar{y}(x) - y_0| < \mu h \longrightarrow (b)$$

if $|x - x_0| \leq |x_1 - x_0|$

b ÷ a we get

$$\frac{|\bar{y}(x_1) - y_0|}{|x_1 - x_0|} = \frac{\mu h}{|x_1 - x_0|} = \frac{\mu h}{h} = \mu$$

$$|\bar{y}(x) - y(x)| = \left| \int_{x_0}^{x} f(t, \bar{y}(t)) - f(t, y(t)) \right|$$

Since $\bar{y}(x)$ & $y(x)$ both lies in R'.

⑤ $|\bar{y}(x) - y(x)| \leq kh \max|\bar{y}(x) - y(x)|$

$\max|\bar{y}(x) - y(x)| \leq kh \max|\bar{y}(x) - y(x)|$

$\max|\bar{y}(x) - y(x)| = 0$, for otherwise we would have $1 \leq kh$ which is Contradiction, to (a), follows that $\bar{y}(x) = y(x)$ for every x in interval $|x - x_0| < h$

Hence proved.