UNIT-V

CRITICAL POINT AND STABILITY FOR LINEAR SYSTEM:

we consider the system

$$\frac{dx}{dt} = a_1 x + b_1 y \quad \text{---} (1)$$

$$\frac{dy}{dt} = a_2 x + b_2 y \quad \text{---} (2)$$

The above system s.T in general a path of one corner a entire phase and denote internet intersect one another.

The only expectation to the stmt occurs of the pts (x_0, y_0) where both

$(a_1 x_0 + b_1 y_0)$ and $(a_2 x_0 + b_2 y_0) = 0$

Such e pt is known as initial path $(a_1 x_0 + b_1 y_0)$ and $(a_2 x_0 + b_2 y_0) = 0$. such a pt as a critical path we assume, $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$

Such that $(0,0)$ is the only critical pt.

W.K.T eqn (1) has a non-trivial soln of the form

$$x = Ae^{mt}$$
$$y = Be^{mt}$$

whenever 'm' is the roots of the quadratic eqn

$$m^2 - (a_1 + b_2)m + (a_1 b_2 - a_2 b_1) = 0 \quad (3)$$

called auxillary eqn of the system

Note that (2) $\Rightarrow$ that zero can't be not of (3)

let m_1 & m_2 be the roots of (3) the initial A $(0,0)$ of system (1) is determined by the natural of member m_1, m_2 is

since we consider a quadratic eqn m in (3)

(i) Roots of m are distinct and real

(ii) Roots of m are real and equal

(iii) Roots are complex and conjugate

Hence deal with the complicated soln we consider the folg five case

case (A)

If the roots m_1 and m_2 are real distinct and of the same sign. Then the critical pt $(0,0)$ is a node.

Proof:

A critical pt is called node we assuming that m_1 & m_2 are both -ve

Let $m_1 < m_2 < 0$

The general soln of the system of linear eqn

$$\left.\begin{aligned} x &= c_1 A_1 e^{m_1 t} + c_2 A_2 e^{m_2 t} \\ y &= c_1 B_1 e^{m_1 t} + c_2 B_2 e^{m_2 t} \end{aligned}\right\} \text{——} (1)$$

Where the A's and B's are definite constants such that $\frac{A_1}{B_1} \neq \frac{A_2}{B_2}$ and c's are arbitrary constant

From (1) $\Rightarrow$ have when $c_2 = 0$ we have

$$\left.\begin{aligned} x &= c_1 A_1 e^{m_1 t} \\ y &= c_1 B_1 e^{m_1 t} \end{aligned}\right\} \text{——} (2)$$

and when $c_1 = 0$ we have

$$\left.\begin{aligned} x &= c_2 A_2 e^{m_2 t} \\ y &= c_2 B_2 e^{m_2 t} \end{aligned}\right\} \text{——} (3)$$

If $c_1 > 0$ the soln of (2) represent a path P is of the following form

$$(2) \Rightarrow e^{m_1 t} = \frac{x}{c_1 A_1} = \frac{y}{c_1 B_1}$$

$$B_1 c_1 x = c_1 A_1 y \Rightarrow B_1 x = A_1 y$$

$$y = \frac{B_1}{A_1} x \Rightarrow y = mx + c$$

i.e., ② represent a path consisting of the half line $A_1 y = B_1 x$ with slope with by B_1/A_1 the form

$$e^{i m_1 t} = \frac{-x}{A_1} = \frac{-y}{B_1}$$

$$= \frac{x}{A_1} + \frac{y}{B_1}$$

In this case soln represent path consisting of the other half of the line

since $m_1 < 0$ both there half line approched $(0,0)$ $t \to \infty$

since $y/x = B_1/A_1$ both enters the straight line $(0,0)$ with slope B_1/A_1

In exactly a same way of the soln of ③ represent 2 half linear pain with slope B_2/A_2 this 2 path approaches $(0,0)$ as $t \to \infty$ and enter limit with

$$y = \frac{B_2}{A_2} \quad \text{slope} \quad m = \frac{B_2}{A_2}$$

The slope $\frac{B_2}{A_2}$ is $c_1 \neq 0$ and $c_2 \neq 0$

The general soln of ① represent path $m_1 < 0$ and $m_2 < 0$. Their paths also approach $(0,0)$ $t \to \infty$ further more since $m_1 < m_2 < 0$.

We have ① $\Rightarrow$ $\dfrac{y}{x} = \dfrac{c_1 B_1 e^{m_1 t} + c_2 B_2 e^{m_2 t}}{c_1 A_1 e^{m_1 t} + c_2 A_2 e^{m_2 t}}$

Dividing Nr and Dr by $c_2 e^{m_2 t}$

$$\frac{y}{x} = \frac{\frac{c_1 B_1 e^{m_1 t}}{c_2 e^{m_2 t}} + B_2}{\frac{c_1 A_1 e^{m_1 t}}{c_2 A_2 e^{m_2 t}} + A_2}$$

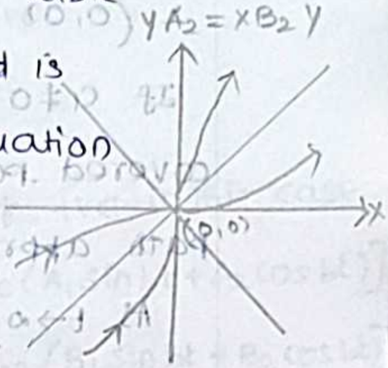
Take,

$$\lim_{t \to \infty} \frac{y}{x} = \frac{\lim_{t \to \infty} \frac{c_1 B_1}{c_2} e^{(m_1 - m_2)t} + B_2}{\lim_{t \to \infty} \frac{c_1 A_1}{c_2} e^{(m_1 - m_2)t} + A_2}$$

AS $t \to \infty$, $y/x = B_2/A_2$, $y = \frac{x B_2}{A_2} \Rightarrow y = mx + c$

All the path enter (0,0) slope B_2/A_2

It is evident that a critical path point is a Node exactly asymptotically stable.



If m_1 & m_2 are both the end is we choose $m_1 > m_2 > 0$. Then the situation is exactly the same except that a paths.

Now approached and enter (0,0) as $t \to \infty$.

Note:

The arrows showing the direction are all reversed it still have a note, but nor its is unstable.

case (B)

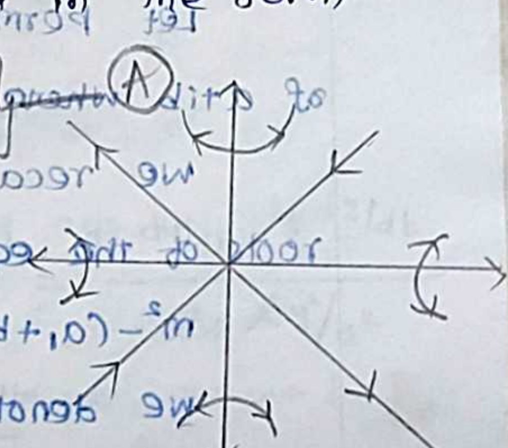
If the roots m_1 & m_2 are real distinct and its opposite sign. Then the critical path (0,0) is a saddle point

Proof:

Given that m_1 & m_2 are real and distinct and opposite sign. So we choose $m_1 < 0 < m_2$.

Then the general soln of the system of can again the expressed in the form

$$\left.\begin{aligned} x &= A_1 c_1 e^{m_1 t} + c_2 A_2 e^{m_2 t} \\ y &= c_1 B_1 e^{m_1 t} + c_2 B_2 e^{m_2 t} \end{aligned}\right\} \quad (A)$$



If we take $c_2 = 0$ then

$$\left.\begin{aligned} x &= c_1 A_1 e^{m_1 t} \\ y &= c_1 B_1 e^{m_1 t} \end{aligned}\right\} \quad (5)$$

If we take $c_1 = 0$ then

$$\left.\begin{aligned} x &= c_2 A_2 e^{m_2 t} \\ y &= c_2 B_2 e^{m_2 t} \end{aligned}\right\} \quad (6)$$

$\therefore m_1 < 0$ the soln of (2) approaches $(0,0)$ as $t \to \infty$

$\therefore m_2 < 0$ the soln of (3) approach $(0,0)$ as $t \to \infty$

$\therefore (0,0)$ is critical path

If $c_1 \neq 0$ & $c_2 \neq 0$. The general soln (1) represents covered path. But since $m_1 < 0 < m_2$ none of these path approaches $(0,0)$ as $t \to \infty$ and $t \to \infty$

As $t \to \infty$ one of those path is asymptotic to one way line path represented by (5) and as $t \to \infty$. Each is asymptotic one of way line path represented by (6) Figure provided two quantiative picture.

In this case the critical pt is a saddle pt and it is obviously unstable.

case (c)

If the roots m_1 & m_2 are conjugate but not pure imaginary then the critical pt $(0,0)$ is a spiral.

Proof :

Given that, the root m_1 & m_2 are complete conjugate. But not pure imaginary.

Let permitted to write m_1, m_2 in the form of $a \pm ib$ where a, b non-vanishing real number

we recall the fact that m_1 & m_2 are roots of the eqn

$$m^2 - (a_1 + b_2)m + (a_1 b_2 - a_2 b_1) = 0$$

we denoted the distinct of (1) as

$$b^2 - 4ac = 0$$

$D = (a_1+b_2)^2 - 4(a_1 b_2 - a_2 b_1)$

$= a_1^2 + b_2^2 + 2b_2 a_1 - 4a_1 b_2 + 4a_2 b_1$

$= a_1^2 + b_2^2 + 4b_1 a_2 - 2a_1 b_2$

$= (a_1 - b_2)^2 + 4a_2 b_1$

We observe that D to be the -ve

i.e., $D < 0$

i.e., $(a_1 - b_2)^2 + 4a_2 b_1 < 0$.

The general soln be system of +ve in this case

$x = e^{at}[c_1(A_1 \cos bt - A_2 \sin bt) + c_2(A_1 \sin bt + A_2 \cos bt)]$

$y = e^{at}[c_1(B_1 \cos bt - B_2 \sin bt) + c_2(B_1 \sin bt + B_2 \cos bt)]$

where A's and B's are define constants

and its c's are arbitrary constants.

(3) $\Rightarrow x \to 0$ and $y \to 0$ as $t \to \infty$

Hence all the paths approaches $(0,0)$ $t \to \infty$ we

now P.T the paths denote enter the pt $(0,0)$ as

$t \to \infty$ but instead c and arrived its a like

manner.

To P.T polar coordinate and P.T along any

$d\theta/dt$ is either +ve, -ve or -ve $\forall\ x \in X$

W.K.T slope, $\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left[\frac{y}{x}\right]$

$$\frac{d\theta}{dt} = \frac{1}{\left[1+\left(\frac{y}{x}\right)^2\right]} \cdot \frac{xy' - yx'}{x^2}$$

$$= \frac{x\, dy/dt - y\, dx/dt}{x^2(1+y^2/x^2)}$$

$$\frac{d\theta}{dt} = \frac{x\, dy/dt - y\, dx/dt}{x^2(1+y^2/x^2)} = \frac{x\, dy/dt - y\, dx/dt}{x^2 + x^2 y^2/x^2}$$

$$\frac{d\theta}{dt} = \frac{x\, dy/dt - y\, dx/dt}{x^2+y^2}$$