ding system of Linear eqn

$$\frac{dx}{dt} = a_1 x + b_1 y; \quad \frac{dy}{dt} = a_2 x + b_2 y$$

sub ⓐ ⓑ in ④ we get

$$\frac{d\theta}{dt} = \frac{x(a_2 x + b_2 y) - y(a_1 x + b_1 y)}{x^2+y^2}$$

$$= \frac{a_2 x^2 + b_2 xy - a_1 xy - b_1 y^2}{x^2+y^2}$$

$$\frac{d\theta}{dt} = \frac{a_2 x^2 + xy(b_2 - a_1) - b_1 y^2}{x^2+y^2}$$

We are interested in only in soln that represent path we assume that

$$x^2+y^2 \neq 0$$

② ⇒ $D = (a_1 - b_2)^2 + 4a_2 b_1 = 0$

The 1^{st} form above is +ve in order that $D<0$, a_1 and b_1 must be opposite signs

We assume $a_1 > 0$; $b_1 < 0$ when $y = 0$

$$\frac{d\theta}{dt} = \frac{a_2 x^2 + 0 - 0}{x^2} = a_2 > 0$$

If $y \neq 0$ then $d\theta/dt$ can't be zero

otherwise $y = a_2 x + (b_2 - b_1) xy - b_1 y^2 = 0$

$\div y^2 \Rightarrow a_2 (x/y^2) + (b_2 - b_1)(x/y) - b_1 = 0$ ∀ real numbers

x/y can't be true

∴ $D<0$ simply the existence of non-real roots which are complex conjugate

⇒ $d\theta/dt > 0$ when $a_2 > 0$

||ly we can prove $d\theta/dt < 0$ when $a_2 > 0$

By eqn ③ $x<y$ change is sign infinitely after as $t \to \infty$ all paths must spiritual into the origin.

Anticlockwise or c-w according as $a_2 > 0$ or $a_2 < 0$. The virtual point in this case is spiral and it is a asympototically stable.

If $a > 0$ the situation is the same expect that the paths approach $(0,0)$ at $t \to \infty$ and the critical point is unstable.

case (D)

If the roots m_1 & m_2 are real and equal then the critical pt $(0,0)$ is a node.

Proof:

Given that roots m_1 & m_2 are equal and real we assume that

$$m_1 = m_2 = m < 0$$

Hence, we arrive at the folg two sub cases.

Subcases:

(i) $a_1 = b_2 \neq 0$ & $a_2 = b_1 = 0$

(ii) All other possibilities leading to a double root of $m^2 - (a_1 + b_2) m + a_1 b_2 - a_2 b_1 = 0$

subcase (i):

we take $a_1 = b_2 \neq 0$

$a_1 = b_2 = a$ (say), $a_2 = b_1 = 0$

then I take up the for m

$$m^2 - (a + a) m + (a \cdot a - 0) = 0$$

$$m^2 - 2am + a^2 = 0$$

$$(m - a)^2 = 0$$

$$m = a, a$$

we result the system of linear eqn

$$\frac{dx}{dt} = a_1 x + b_1 y, \quad \frac{dy}{dt} = a_2 x + b_2 y$$

Introducing $a_2 = b_1 = 0$, we have

$$\frac{dx}{dt} = a_1 x, \quad \frac{dy}{dt} = b_2 y$$

$\int$ ing

$$\int \frac{dx}{x} = a_1 \int dt, \quad \int \frac{dy}{y} = b_2 \int dt$$

$$\log x = a_1 t + \log c_1 \qquad \log y = b_2 t + \log c_2$$

$$\log\left(\frac{x}{c_1}\right) = a_1 t \qquad \log\left(\frac{y}{c_2}\right) = b_2 t$$

$$\frac{x}{c_1} = e^{a_1 t} \qquad \frac{y}{c_2} = e^{b_2 t}$$

$$x = c_1 e^{a_1 t} \qquad y = c_2 e^{b_2 t}$$

$$\frac{x}{c_1} = \frac{y}{c_2} \Rightarrow \frac{y}{x} = \frac{c_2}{c_1} \Rightarrow y = \frac{c_2}{c_1} x$$

Here the path defined

Half line of all possible slope figure

since $m<0$; each path approaches and entries $(0,0)$ as $t \to \infty$

The critical pt is therefore a node and it is asymptotically stable. if $m>0$. we have the same situation except stable that the path enter $(0,0)$ as $t \to \infty$

Hence the arrows in the figure are reversed and $(0,0)$ is unstable.

Subcase (ii):

we now consider the subcase (ii) the general soln of system of Linear eqn can be expressed in the form.

$$\left.\begin{aligned} x &= c_1 A e^{mt} + c_2 (A_1 + A_2 t) e^{mt} \\ y &= c_1 B e^{mt} + c_2 (B_1 + B_2 t) e^{mt} \end{aligned}\right\} \quad (1)$$

when A's and B's are define constant and c's are arbitrary constant $c_2 = 0$

$$① \Rightarrow x = c_1 A e^{mt}, \quad y = c_1 B e^{mt}$$

$$② \Rightarrow \frac{x}{c_1 A} = e^{mt}, \quad \frac{y}{c_1 B} = e^{mt}$$

$$\frac{x}{c_1 A} = \frac{y}{c_1 B} = e^{mt}$$

$$\frac{x}{A} = \frac{y}{B} \Rightarrow Bx = Ay \Rightarrow y = \frac{B}{A} x$$

② ⇒ That there soln represented to way line path using on the line ~~By = By~~ $Ay = Bx$

with slope B/A & since $m<0$ both path approach $(0,0)$ as $t \to \infty$ in fig(i)

If $c_2 = 0$, the soln represents & since $m<0$ these path approaches $(0,0)$ as $t \to \infty$.

Now,

$$\frac{y}{x} = \frac{c_1 B e^{mt} + c_2 (B_1 + B_2 t) e^{mt}}{c_1 A e^{mt} + c_2 (A_1 + A_2 t) e^{mt}}$$

$$= \frac{c_1 B + c_2 (B_1 + B_2 t)}{c_1 A + c_2 (A_1 + A_2 t)}$$

Taking $\lim t \to \infty$

$$\lim_{t \to \infty} \frac{y}{x} = \lim_{t \to \infty} \frac{\frac{c_1 B}{c_2} + (B_1 + B_2) t}{\frac{c_1 A}{c_2} + (A_1 + A_2) t}$$

$$= \lim_{t \to \infty} \frac{\frac{c_1 B}{c_2} + \frac{B_1}{t} + B}{\frac{c_1 A}{c_2} + \frac{A_1}{t} + A} = \frac{B}{A}$$

As $t \to \infty$ the slope $\frac{dy}{dx}$ approaches B/A. so that curved path a/c enter $(0,0)$ with slope B/A

We further notice that $y/x \to B/A$ as $t \to \infty$ fig(v) gives that the paths it is clear that $(0,0)$ is noted that is asymptotically stable.

If $m > 0$ the situation in unchanged except that direction of the paths are reversed & the critical point is unstable.

Case (E):

If the roots m_1 & m_2 are pure imaginary then the critical pt (0,0) in the center.

Proof:

Already in case C.

We have the roots are complex conjugate and in the form $a+ib$

It is enough if two we consider the same case with $a=0$ & $b \neq 0$

Hence the general soln of ① is given by

$$x = c_1 [A_1 \cos bt - A_2 \sin bt] + c_2 [A_1 \sin bt + A_2 \cos bt]$$

$$y = c_1 [B_1 \cos bt - B_2 \sin bt] + c_2 [B_1 \sin bt + B_2 \cos bt]$$

$x(t)$ & $y(t)$ non-periodic & each is closed curve is arrounding the origin.

fig (v) exhibits the fact that curves are actually ellipse.

This be proved by solving the differentiating each of path.

$$\frac{dy}{dx} = \frac{a_2 x + b_2 y}{a_1 x + b_1 y}$$

our critical (0,0) is evidentaly a center that is stable but not asymptotically

stability by Liaduno's direct method

the system of form

$$\frac{dx}{dt} = F(x, y)$$

$$\frac{dy}{dt} = G(x, y)$$

where F and G are cts and has cts partial derivatives throughout the plane

A system of this kind in which the independent variable 't' doesn't appears in F and G is said to be autonomous system by ①.

We consider the autonomous system and assume that such a system has an isolated critical pt and w log we take, to be the origin (0, 0).

we shall to fact to be then as isolated critical pt in (x_0, y_0).

If E is a circle centre on (x_0, y_0) then contain no other critical pts.

Let $C = [x(t), y(t)]$ be a paths of ① and consider a function $E(x, y)$ that is cts & has 1st partial derivatives in region contrading this paths.

If a pt (x, y) moves along the path in accordance has the eqn's

$x = x(t)$, $y = y(t)$.

Then,

$E(x, y)$ can be regarded as a function of t along C. we denoted this function at $E(t)$.

Then its rate of change in terms of total difference of $\frac{dE}{dt} = \frac{dE}{dx} \cdot \frac{dx}{dt} + \frac{dE}{dy} \cdot \frac{dy}{dt}$

$$\frac{\partial E}{\partial t} = \frac{\partial E}{\partial x} \cdot F + \frac{\partial E}{\partial y} \cdot G$$

THEOREM:

The critical pt (0,0) of the linear system

$\frac{dx}{dt} = a_1 x + b_1 y$; $\frac{dy}{dt} = a_2 x + b_2 y$; is asymptotically stable

iff the co.eff $p = -(a_1 + b_2)$; $q = a_1 b_2 - b_1 a_2$ of the

auxillary eqn $m^2 - (a_1 + b_2) m + (a_1 b_2 - b_1 a_2)$ are both +ve

Proof:

We are eqn that

$$m^2 - (a_1 + b_2) m + (a_1 b_2 - b_1 a_2) = 0$$

If m_1, m_2 are the roots of the eqn (1) then (2) can be expressed $a_0(m - m_1)(m - m_2) = 0$

i.e., $m^2 - (\text{sum of roots}) m + \text{Pro of roots} = 0$

$$m^2 + Pm + q = 0$$

where $P = -(m_1 + m_2)$; $q = m_1 m_2$

Then the earlier +ve case can be expressed in terms of P, q rather than in terms of m_1 & m_2 more over we can intercept (1)

These case is the pq plane which is illustration in the folg. plane.

Then we write arrive at a striking diagram that displace at the glance the nature and stability of the critical pt (0,0) the expansion.

$\frac{dE}{dt} = \frac{dE}{dx} \cdot F + \frac{dE}{dy} \cdot G$ at the neat of Liapunov's formula.

We know provides the definition which are useful in the square.