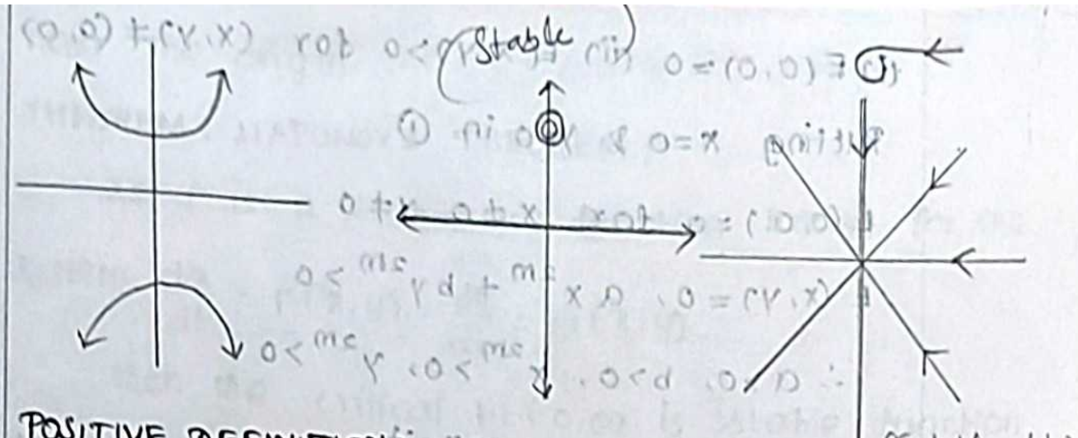


POSITIVE DEFINITION:

We suppose that $E(x,y)$ is cts and has cts 1^{st} partial derivatives in some region containing origin. We say that the function $E(x,y)$ is +ve define if (i) $E(0,0)=0$ (ii) $E(x,y)>0$ for $(x,y)\neq(0,0)$.

NEGATIVE ~~DIRECTION~~: Definite

We suppose that $E(x,y)$ is cts and has cts 1^{st} order partial derivatives in some region containing $(0,0)$ we say that the function $E(x,y)$ is defined by

(i) $E(0,0)=0$ (ii) $E(x,y)=0$ for $(x,y)\neq(0,0)$

POSITIVE SEMI-DEFINITE:

We suppose that $E(x,y)$ is cts and processing cts partial derivative on some region containing $(0,0)$ We say that the function $E(x,y)$ in -ve semi definite if

$$E(x,y)\leq 0 \text{ for } (x,y)\neq(0,0)$$

ILLUSTRATION:

consider $E(x,y)=ax^{2m}+by^{2n}$ where a,b are +ve constant & m,n are +ve integers,

P.T E is +ve definite.

soln:

Given $E(x,y)=ax^{2m}+by^{2n}$.

To P.T E is +ve definite if it is suffices to proves

(i) $E(0,0)=0$ (ii) $E(x,y)>0$ for $(x,y)\neq(0,0)$

Putting $x=0$ & $y=0$ in ①

$E(0,0)=0$ for $x\neq 0, y\neq 0$

$E(x,y)=0,\ ax^{2m}+by^{2m}>0$

$\therefore a>0,\ b>0,\ x^{2m}>0,\ y^{2m}>0$

$E(x,y)$ is +ve definite

Remark:

$E(x,y)$ is -ve definite $\Leftrightarrow -E(x,y)$ is +ve definite

ILLUSTRATION: Let $E(x,y)=ax^{2m}+by^{2n}$, $E(x,y)$ is -ve definite

Proof: Given $E(x,y)=ax^{2m}+by^{2m}$

To prove $E(x,y)$ is -ve definite

i.e., $E(x,y)>0$ for $(x,y)\neq(0,0)$

Put $x\neq 0, y\neq 0$ in ① we get

$E(x,y)=ax^{2m}+by^{2m}<0$

$\therefore a<0,\ b<0$

$E(x,y)<0$

$\therefore E(x,y)$ is -ve definite

Remark:

Then function x^{2m}, y^{2n} & $(x,y)^{2m}$ aren't +ve definite but are +ve semi-definite

LIAPUNOV'S FUNCTION:

Let $E(x,y)$ be function $\ni$

(i) $E(0,0)=0$; (ii) $E(x,y)>0$ for $E(x,y)\neq(0,0)$

i.e., $E(x,y)$ is a +ve function we say that $E(x,y)$ is a Liapunov's function.

If $\dfrac{dE}{dt}=\dfrac{dE}{dx}\cdot F+\dfrac{dE}{dy}\cdot G$ is -ve semi definite

$\dfrac{dE}{dx}\cdot F+\dfrac{dE}{dy}\cdot G<0$ for $(x,y)\neq(0,0)$

i.e., $\dfrac{dE}{dt}<0$, E is decreasing along the path of ① near the origin.

THEOREM : LIAPUNOV'S THEOREM :

If f is a Liapunov's function $E(x,y)$ for the system $\frac{dx}{dt} = F(x,y)$, $\frac{dy}{dt} = G(x,y)$

Then the critical pt $(0,0)$ is stable function more if it this function has the $\frac{\partial E}{\partial x} \cdot F + \frac{\partial E}{\partial y} G$ is -ve definite. Then the critical pt $(0,0)$ is asymptotically stable.

Proof :

Let C_1 be the circle of radius $R>0$ when R is a +ve quantity.

C is connected on the origin

We also assume that C_1 is small enough to line entirely in the domain of the definition of the Function E.

Since $E(x,y)$ is constant & +ve definite. It has a +ve minimum 'm' on G_1, $E(x,y)$ is a constant at the origin & vanish whenever (x,y) is inside the circle C_2 of radius.

Now let $C = \{x(t)\}$ be any path which is inside C_2 for $t = t_0$

$$\frac{dE}{dt} = \frac{dE}{dx} \cdot F + \frac{dE}{dy} \cdot G \text{ is a +ve semi definite}$$

$$\text{i.e., } \frac{dE}{dt} < 0$$

$E(t) \leq E(t_0) < m$ for $t < t_0$ It follows that the path C can never reach the C_1 for any $t > t_0$.

case (ii)

Hence the critical path p/s $(0,0)$ stable. To

Prove 2^{nd} proof of the theorem.

It sufficient to P.T

$\frac{dE}{dt} = \frac{dE}{dx} \cdot F + \frac{dE}{dy}$ G is a -ve semi definite. the critical path pt (0,0) is -ve semi definite.

if $\frac{dE}{dt} \le 0$

It follows that $E(t)$ is decreasing function. $E(t) \to 0$ [∵ $E(x,y)$ is +ve definite the path C approaches then critical pt (0,0)]

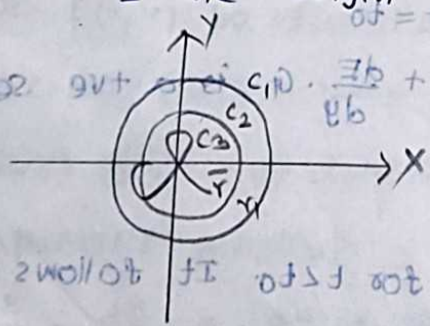
In otherword E(t) has bounded below by zero. we conclude that,

E(t) approaches some lt $L \ge 0$ as $t \to \infty$.

To prove : $E(t) \to 0$ we need to prove $L=0$ we assume that $L>0$ and arrive at a contradictio

we choose number $\bar{r} < r$. The property then $E(x,y) < L$ whenever (x,y) is inside circle C with radius $\bar{r}$.

since $\frac{dE}{dt}$ is cts & -ve definite it has a -ve maximum $(-K)$ in the ring consisting of the circles C_1 & C_2 the origin b/w them



The ring contains the entire path C for $t \ge t_0$, so that the eqn

$$E(t) = E(t_0) + \int_{t_0}^{t} \frac{dE}{dt} \cdot dt$$

$= E(t_0) + \int_{t_0}^{t} dE$

$E(t) = E(t_0) - K + E(t) - E(t_0)$

$\leq E(t_0) - K(t - t_0) \quad \forall \; t \geq t_0$

$E(t) \leq E(t_0) - K(t - t_0)$

lie in $t \to \infty$ the R.H.S of ① approaches $-\infty$

$E(t) \to -\infty$ as $t \to \infty$

which is the contradiction on fact that $E(x,y) \geq 0$;

$t = 0$

Hence the theorem.

THEOREM: Ex:

consider the eqn of motion of mass 'm' attached to spring,

$$m \frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

Here $c \geq 0$ is a constant representing the viscosity of the medium through which the mass moves & $k > 0$ is the spring constant.

Proof:

The Autonomous system equivalent to eqn ①

$$\frac{dx}{dt} = y; \; \frac{dy}{dt} = -\frac{kx}{m} - \frac{cy}{m}$$

① $\Rightarrow \frac{m\,dy}{dt} + cy + kx = 0$

$$\frac{m\,dy}{dt} = -kx - cy$$

$$\frac{dy}{dt} = -\frac{k}{m}x - \frac{c}{m}y$$

Hence the autonomous system for ① is

$$\frac{dx}{dt} = y, \; \frac{dy}{dt} = -\frac{k}{m}x - \frac{cy}{m}$$

The only critical pt of ② is $(0,0)$

The Kinetic Energy of the mass is $\frac{1}{2}my^2$