potential energy stored in the spring

$\therefore KE = \frac{1}{2} m v^2$

$$\int_0^x K \cdot x \, dx = K\left(\frac{x^2}{2}\right)_0^x = \frac{Kx^2}{2}$$

Total energy = KE + PE

$$= \frac{1}{2} m y^2 + \frac{1}{2} K x^2$$

we observe $E(0,0) = 0$ and for $(x \neq 0, y \neq 0)$

$E(x,y) > 0$ $E(x,y)$ is +ve definite

$$E(x,y) = \frac{1}{2} m y^2 + \frac{1}{2} K x^2$$

$$\frac{dE}{dt} = \frac{dE}{dx} F + \frac{dE}{dy} G$$

where $F = y$ $G = -\frac{K}{m} x - \frac{c}{m} y$

Diff w.r.to "x"

$$F = \frac{\partial F}{\partial x} = \frac{1}{2} 2xK = xK$$

$$G = \frac{\partial E}{\partial y} = \frac{1}{2} 2my = my$$

$$\frac{\partial E}{\partial t} = Kxy + my\left[\frac{-K}{m} x - \frac{c}{m} y\right]$$

$$= Kxy + my\left(\frac{-K}{m}\right)x - my\left(\frac{c}{m}\right)y$$

$$= Kxy - Kxy - cy^2$$

$$\frac{\partial E}{\partial t} = -cy^2$$

$$\frac{\partial E}{\partial t} \leq 0$$

$\therefore \frac{\partial E}{\partial t}$ is -ve semi definite

Eqn ③ is a Liapunov's function for ② and the critical pt (0,0) is stable.

EXAMPLE

The system $\frac{dx}{dt} = -2xy$, $\frac{dy}{dt} = x^2 - y^3$ has (0,0) as an isolated critical pt. we now try to prove the stability by constructing a Liapunov's function is of form $E(x,y) = ax^{2m} + by^{2n}$

$$\frac{\partial E}{\partial x} = 2x^{2m-1}(am) = 2amx^{2m-1}$$

$$\frac{\partial E}{\partial y} = 2n\,by^{2n-1}$$

Proof:

Given that $\frac{dx}{dt} = -2xy$ [$\because F(x,y) = -2xy$]

$\frac{dy}{dt} = x^2 - y^3$ [$\because G(x,y) = -x^2 - y^3$]

$$\frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G = 0$$

$$2amx^{2m-1}(-2xy) + 2bny^{2n-1}(x^2 - y^3)$$

$$\Rightarrow -4amx^{2m}y + 2bny^{2n-1}x^2 - 2bny^{2n+2}$$

we now the express bn in parenthesis vanish and inspection s.T this can be denoted by choosing $m=1$ $n=1$ $a=1$ $b=2$ with these choice we have

$E(x,y) = x^2 + 2y^2$

$E(0,0) = 0$

$E(x,y) > 0$ for $(x,y) \neq 0$

$E(x,y)$ is +ve definite

$E(x,y) = x^2 + 2y^2$

$$\frac{\partial E}{\partial x} = 2x, \quad \frac{\partial E}{\partial y} = 4y$$

$$\frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G = 2x(-2xy) + 4y(x^2 - y^3)$$

$$= -4x^2y + 4x^2y - 4y^4$$

$$= -4y^4$$

$\frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G$ is -ve semi definite

THEOREM

The function $E(x,y) = ax^2 + byx + cy^2$ is +ve definite $\Leftrightarrow$ $a>0$, $b^2 - 4a < 0$ and is +ve definite $\Leftarrow$

$a>0$ & $b^2-4ac<0$.

Proof:

we are given that

$$E(x,y) = ax^2 + byx + cy^2$$

If $y=0$, $E(x,0) = ax^2$

$E(x,0)>0$ for $x \neq 0 \Leftrightarrow a>0$.

If $y \neq 0$ we have

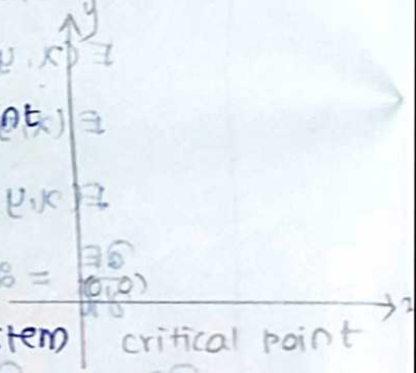
$$E(x,y) = bxy + cy^2 + ax^2$$
$$= y^2\left(\frac{bx}{y} + c + \frac{ax^2}{y^2}\right)$$
$$= y^2\left[b\left(\frac{x}{y}\right) + a\left(\frac{x^2}{y^2}\right) + c\right]$$
$$= y^2\left[a\left(\frac{x^2}{y^2}\right) + b\left(\frac{x}{y}\right) + c\right]$$

when $a>0$ the bracket play on (x,y) is +ve $\forall (x,y) \Leftrightarrow b^2-4ac<0$

Thus prove the 1st parts of the thm the 2nd part of the thm is obtained by taking the function $E(x,y)$

SIMPLE CRITICAL POINTS ON NON-LINEAR SYSTEM:

In this section we consider the restriction upon a critical point $(0,0)$ in order that its simple critical point.



consider an autonomous system

$$\frac{dx}{dt} = F(x,y), \quad \frac{dy}{dt} = G(x,y)$$

with an isolated point of $(0,0)$ if $F(x,y)$ and $G(x,y)$ can be expressed its power series from x and y

$$\frac{dx}{dt} = a_1x + b_1y + c_1x^2 + d_1xy + c_1y^2 + \cdots$$

$$\frac{dy}{dt} = a_2x + b_2y + c_2x^2 + d_2xy + c_2y^2 + \cdots$$

when (x) and (y) are small

i.e., when (x, y) are close to the origin

(∴ (0,0) (x^2, y^2) etc are still small which observe neglected.

We discard the non-linear form and conjecture and that the qualitative behaviour of the path ② near the critical point (0,0) to the path of the linear system

$$\frac{dx}{dt} = a_1x + b_1y, \quad \frac{dy}{dt} = a_2x + b_2y$$

We first recall, what do you mean by a stable critical point

The critical pt is said to be stable it for each +ve number R such that,

a +ve number

Every path which is inside the circle $x^2+y^2=r^2$ by definition for some $t=t_0$ remain inside the circle $x^2+y^2=R^2$ ∀ $t>t_0$

We also recall the assumption stable critical pt. The critical path is said to be asymptotically stable. If it is stable and there exist a circle $x^2+y^2=r^2$.

such that every path which is invisible the origin as $t\to\infty$ finally if our critical pt stable then it is called unstable;

EXAMPLE:

To find the simple critical pt

9.
Nov-16

$$\frac{dx}{dt} = -2x + 3y + xy; \quad \frac{dy}{dt} = -x + y - 2xy^2$$

10.
Nov-16

Similar method

Imp. ?

→ In this case of the system

$$\frac{dx}{dt} = -2x+3y+xy \quad \text{---} ①$$

$$\frac{dy}{dt} = -x+y-2xy^2 \quad \text{---} ②$$

$a_1 = -2, \; b_1 = 3$

$b_2 = 1$

Now, $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$

we have,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} -2 & 3 \\ -1 & 1 \end{vmatrix} = -1 \neq 0$$

$-2(r\cos\theta)+3(r\sin\theta) + r^2\cos\theta\sin\theta$

$-2r\cos\theta + 3r\sin\theta$

so ② is satisfied

Furthermore, for using polar coordinates $+ r^2\cos\theta\sin\theta$

$x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2 = r^2$

we see that,

$$\frac{|f(x,y)|}{\sqrt{x^2+y^2}} = \left|\frac{r^2\sin\theta\cos\theta}{r}\right| \leq r \text{ and}$$

$-r\cos\theta + r\sin\theta$

$$\frac{|g(x,y)|}{\sqrt{x^2+y^2}} = \left|\frac{2r^3\sin^2\theta\cos\theta}{r}\right| \leq 2r^2$$

$-2(r\cos\theta)(r^2\sin^2\theta)$

so $\frac{f(x,y)}{r}$ and $\frac{g(x,y)}{r} \to 0$ as $(x,y) \to (0,0)$

this s.t. cts

$$\lim \frac{f(x,y)}{\sqrt{x^2+y^2}} = 0 \text{ and } \lim \frac{g(x,y)}{\sqrt{x^2+y^2}} = 0$$

are also satisfied so $(0,0)$ is a simple critical pt of the system

$$\frac{dx}{dt} = -2x+3y+xy, \quad \frac{dy}{dt} = -x+y-2xy^2$$

THEOREM

Let $(0,0)$ be a simple critical point of the non-linear system

$$\frac{dx}{dt} = a_1x + b_1y + f(x,y)$$

$$\frac{dy}{dt} = a_2x + b_2y + g(x,y)$$ and consider the

related linear system $\frac{dx}{dt} = a_1x + b_1y$, $\frac{dy}{dt} = a_2x + b_2y$

If the critical pt $(0,0)$ of $\frac{dx}{dt} = a_1x + b_1y$, $\frac{dy}{dt} = a_2x + b_2y$ asymptotically stable the critical pt $(0,0)$ of $\frac{dx}{dt} = a_1x + b_1y + f(x,y)$, $\frac{dy}{dt} = a_2x + b_2y + g(x,y)$ also asymptotically stable.

Proof :

W.K.T

The coefficient of the linear system

$$\frac{dx}{dt} = a_1x + b_1y, \quad \frac{dy}{dt} = a_2x + b_2y$$

satisfied the partition.

$$p = (-a_1 + b_2) > 0 \text{ and } q = a_1b_2 - a_2b_1 > 0$$

Now define $E(x,y) = \frac{1}{2}(ax^2 + 2bxy + cy^2)$ ——— ①

$$a = \frac{a_2^2 + b_1^2 + (a_1b_2 - a_2b_1)}{\partial}$$

$$b = \frac{-a_1a_2 + b_1b_2}{\partial}$$

$$c = \frac{a_1^2 + b_1^2 + a_1b_2 - a_2b_1}{\partial}$$

where $\partial = pq = -(a_1 + b_2)(a_1b_2 - a_2b_1)$ by ①

we have that $\partial > 0$ and $a > 0$. Also an easy calculate s.t $\partial(ab - b^2) > 0$

$$\partial(ab - b^2) = \{a_2^2 + b_2^2 + a_1^2 + b_1^2 + a_2^2 + b_2^2 + a_1^2 + b_1^2\}$$
$$(a_1b_2 - a_2b_1) + (a_1b_2 - a_2b_1)^2 - (a_1a_2 - b_1b_2)$$

$$= (a_2^2 + b_2^2 + a_1^2 + b_1^2)(a_1b_2 - a_2b_1) + 2(b_2a_1 - a$$

$$> 0$$

So $b^2 - 4ac < 0$

Thus by known thm W.K.T the function $F(x,y$ is +ve definite another calculation

$$\frac{\partial E}{\partial x}(a_1x+b_1y)+\frac{\partial E}{\partial y}(a_2x+b_2y)=-(x^2+y^2) \quad (3)$$

This function is clearly -ve definite to $E(x,y)$ is Liapunov's function for the linear system

$$\frac{dx}{dt}=a_1x+b_1y \;,\; \frac{dy}{dt}=a_2x+b_2y$$

we now P.T $E(x,y)$ is also a Liapunov's function for the non-linear system

$$\frac{dx}{dt}=a_1x+b_1y+f(x,y) \;;\; \frac{dy}{dt}=a_2x+b_2y+g(x,y)$$

If f and g are defined by

$$F(x,y)=a_1x+b_1y+f(x,y)$$

$$G(x,y)=a_2x+b_2y+g(x,y)$$

(1) Then E is known to be +ve definite it suffices to S.T $\frac{\partial E}{\partial x}\cdot F+\frac{\partial E}{\partial y}\cdot G$ is -ve definite (5)

If we are 2 and 3 becomes,

$-(x^2+y^2)+ax+by\,f(x,y)+bx+cy\,g(x,y)$ & by introducing polar coordinate we can wrote this

$$-x^2+x[a\cos\theta+b\sin\theta\, f(x,y)+bx+cy\, g(x,y)]$$

Denoted the largest of the number

$|a|\,|b|\,|c|$ by k our assumption that (1) now implies that

$|f(x,y)|<\gamma\,|bx|$ &

$|g(x,y)|<\gamma\,|bk|$, $\forall$ sufficient small $\gamma>0$

$$\text{so, } \frac{\partial E}{\partial x}F+\frac{\partial E}{\partial y}G<-\gamma^2+\frac{4x\gamma^2}{8k}=\frac{-\gamma^2}{3}$$

For this γ's thus $F(x,y)$ is a +ve define function with that (8) is +ve definite

By known thm,

Now implies that $(0,0)$ is asymptotically stable critical pt of ④

This proof complete

Proof:

A non-trivial soln $y(x)$ of the eqn

$$y'' + \lambda(y) = 0 \quad \text{———} \quad ①$$

that satisfied a boundary condition

$$y(0) = 0, \quad y(\pi) = 0 \quad \text{———} \quad ②$$

There we wish to satisfy one condition at each of two distinct value of x problem of this kind are called boundary value problem.

If λ is $(-ve)$ tells at that only the trivial soln of ① can satisfy ②

If $\lambda = 0$

The general soln of ① is

$$y(x) = c_1 x + c_2 x$$

If λ is $+ve$ where the general soln of ① is

$$y(x) = c_1 \sin\sqrt{\lambda x} + c_2 \cos\sqrt{\lambda x}$$

$$y(x) = y_0, \quad y(0) = 0$$

$$y(0) = c_1 \sin\sqrt{\lambda(0)} + c_2 \cos\sqrt{x(0)}$$

$$y(0) = c_2 \cos\sqrt{\lambda}$$

$$0 = c_2 \cos\sqrt{\lambda}$$

$$y(x) = c_1 \sin\sqrt{\lambda(x)} + 0$$

& since $y(0)$ must so this reduced to

$y_1(x) = c_1 \sin\sqrt{\lambda}(x)$

If our prblm has a soln it must be of the form

③ for the second boundary condition $\lambda(\pi)=0$.

It is clear that $\sqrt{\lambda}\pi$ must be equal to $n\pi$

for some +ve integer n. so $\lambda = n^2$

λ must equal one of the number 1, 4, 9, ..

Those value λ are called eigen value of the

prbm and corresponding soln

$\sin x, \sin 2x, ..$ are called eigen function.

4/9/2019

UNIT-I

THEOREM:

Let x_0 be an ordinary pt of the differential eqn. $y'' + P(x)y' + Q(x)y = 0$ and let a_0 be the arbitrary constant of the unique function $y(x)$.

i.e, Analytic x_0 is a soln of ① in a certain noted of this pt and satisfied the initial condition $y(x_0) = a_0$, $y'(x_0) = a_1$.

Proof:

$$y'' + P(x)y' + Q(x)y = 0 \quad \text{——— ①}$$

For the sake of convenience, we restrict our arguement to the case in which $x_0 > 0$. This permits us to work with power series in x rather than $x - x_0$, and involves no loss of generality. With this slight simplification, the hypothesis of the theorem is that $P(x)$ and $Q(x)$ are analytic