UNIT - IV

Oscillation and sturm seperation theorem

Theorem :

If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of $y''+P(x)y'+Q(x)y=0$. Then the zero's of this functions are distinct and occur alternatively in the sense that $y_1(x)$ vanishes exactly once b/w any two successive zero's of $y_2(x)$ & conversly.

proof Given $y_1(x)$ and $y_2(x)$ are LI soln of

$$y''+p(x)y'+Q(x)y=0$$

since y_1 and y_2 are LI their wronkskain

$$W(y_1,y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0.$$

$$y_1(x)y_2'(x) - y_2(x)y_1'(x) \neq 0.$$

The zeros of y_1 and y_2 must be distinct otherwise.

$$W(y_1,y_2) = 0$$

$$y_1(x)y_2'(x) - y_2(x)y_1'(x) = 0$$

If we take $y_2(x)_0 = 0$ replains x by x_0 we have p if $x=x_0$.

$$y_1(x_0)y_2'(x_0) - y_2(x_0)y_1'(x_0) = 0$$

$$y_1(x_0)y_2'(x_0) \Rightarrow y_1(x_0) = 0.$$

we have assume that x_1, x_2 are successive zero of y_2 and P.T y_1 vanishes b/w thesepoints by hypothesis is

$$y_2(x_0)=0$$

$$y_2(x_1)=0 \text{ \& } y_2(x_2)=0$$

Now, $W(y_1,y_2) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$

at $x=x_1$, $x=x_2$

In general $y_2(x) = 0$

$w(y_1, y_2) = y_1(x) y_2'(x)$ at $x = x_1$ & $x = x_2$

since $w(y_1, y_2) \neq 0$

$y_1(x) \neq 0$ & $y_2'(x) \neq 0$ at $x = x_1$ and x_2

Let, $y = f(x)$ $\begin{cases} \text{if } dy/dx > 0, \text{ then } y \text{ is an increasing} \\ \text{if } dy/dx < 0, \text{ then } y \text{ is an decreasing} \end{cases}$

furthermore $y_2'(x_1)$ & $y_2'(x_2)$ must has opposite signs. If $y_2(x)$ is increasing at x_1 must have decreasing at x_2 and vice versa.

Since the wronkskain Constant sign $y_1(x_1)$ & $y_2(x_2)$ must also have opposite sign and therefore $y_1(x)$ must vanishes at same point b/w x_1 & x_2

Note that y_1 can't vanishes more than once b/w x_1 and x_2 otherwise by the same argument must vanishes b/w the zero of y. which is our assumption that x_1 & x_2 are successive zero's

Theorem:

If $q(x) < 0$ and if $u(x)$ is non-trivial soln of $u'' + q(x)$, $u(x) \neq 0$. Then $u(x)$ has atmost one zero.

proof Let x_0 be a zero's of $u(x)$ then $u(x_0) = 0$ we say that

$u(x)$ is a non trivial if $u(x_0) \neq 0$, $y'(x_0) \neq 0$.

we assume that,

$u'(x) > 0$ then $u(x)$ is the over some into the right of x_0.

Given, $q(x) = 0$, then

$$u''(x) + q(x)\,u(x) = 0$$

$$u''(x) = -q(x)u(x)$$

$$u''(x) > 0$$

let a be a function on the same interval. $u'(x)$ is an increasing function.

$u(x)$ Can't have a zero to the first and some to the last of x_0

$u'(x_0)$ & by similar argument, then prove $u(x)$ has either no zero's at all or only one.

$\therefore$ $u(x)$ has atmost one zero.

Theorem

Let $u(x)$ be any non-trivial soln of $u''+q(x)\,u(x)=0$. where $q(x)>0$, $\forall x>0$. If $\int^{\infty} q(x)\,dx=\infty$. Then $u(x)$ has infinitely many zero's on the +ve x axis.

Proof Given that, $u(x)$ be any non-trivial soln of $u''+q(x)u(x)=0 \longrightarrow (1)$

where, $q(x)\geq 0$ $\forall x>0$

To P.T, $u(x)$ has ∞ many zero's on the +ve axis.

$$\int^{\infty} q(x)(dx) = \infty$$

To prove theorem, we assume that the Contradiction. we assume $u(x)$ vanishes atmost. A finite number of times for $0<x<x$ for every a point $x_0>0$.

To S.T. $u(x_0)\neq 0$ $\forall$ $x\geq x_0$

with oat any loss of generality assume $u(x)\geq 0$ $\forall$ $x>x_0$

$u(x)$ Can be replaced by its -ive it necessary our object is accomplished If we established a Contradiction

we now S.T,

$u'(x)$ is -ve.

If we put,

$$V(x) = \frac{-u'(x)}{u(x)} \quad \text{for} \quad x\geq x_0$$

$$V'(x) = \frac{-u(x)u''(x) - u'(x)\,u'(x)}{[u(x)]^2}$$

$$= -\frac{uu''+u'^2}{u^2}$$

$$= q(x)+v^2(x)$$

on integrating ~~$x>x_0$~~ $x > x_0$

we p.T $u(x)$ & $u'(x)$ have opposite signs it approches ∞.

$u(x)$ is $-$ve which is a $\Rightarrow\Leftarrow$

Hence,

$u(x)$ has ∞ many zero's on the x axis.

Sturm Comparision Test :

Let $y(x)$ be a non trivial soln of
$y''+q(x)y=0 \longrightarrow (1)$ where $q(x)$ is a +ve function on $[a,b]$. Then has atmost a finite number of zero's in the interval.

proof: Given $y''+q(x)y=0$

$q(x)$ is non trivial soln of $u''+q(x)u(x)=0$

$q(x)>0$..

we assume that $y(x)$ is infinite number of zero's in $[a,b]$

Then $\exists$ a point x_0 in $[a,b]$

$$v = -\frac{u'(x)}{u(x)}. \text{ Then,}$$

let us assume,

$$v' = -\left[\frac{uu''-u'u'}{u^2}\right]$$

$$= -\left[\frac{uu''-u'^2}{u^2}\right]$$

$$= -\frac{\cancel{u}u''}{u^{\cancel{2}}} + \frac{u'^2}{u^2}$$

$$v'(x) = q(x)+v^2. \longrightarrow (2)$$

Now integrating from x_0 to x whose $x>0$

$$\int_{x_0}^{x} v'(x)\,dx = \int_{x_0}^{x} q(x)\,dx + \int_{x_0}^{x} v^2\,dx$$

$$\int_{x_0}^{x} dv = \int_{x_0}^{x} q(x)\,dx + \int_{x_0}^{x} v^2\,dx$$

$$[v]_{x_0}^{x} = \int_{x_0}^{x} q(x)\,dx + \int_{x_0}^{x} v^2\,dx$$

By substitute

$$\int_{1}^{\infty} q(x)\,dx = \infty$$

$$v(x) - v(x_0) = \infty + \int_{x_0}^{x} v^2\,dx.$$

$v(x)$ is +ve if x is bounded to approach. A sequence of zero's $x_n \neq x_0$ $\exists$: $x_n \to x_0$ Since $y(x)$ is continuous we have,

$$y(x_0) = \lim_{x_n \to x_0} y(x)$$

$$y'(x_0) = \lim_{x_n \to x_0} \frac{y(x_n) - y(x_0)}{x_n - x_0}$$

The above two statement implies that $y(x)$ has trivial soln of (1).

Hence $y(x)$ has atmost finite numbers zero's in the interval.

Theorem.

Let $y(x)$ & $z(x)$ be non trivial soln $y''(x) + q(x)\,y(x) = 0$ and $z''(x) + y(x)\,z(x) = 0$ where $q(x)$ & $v(x)$ are the function $\exists : q(x) > 0$, Then $y(x)$ vanishes atleast one b/w two successive zero's of z

Given that

$$y''(x) + q(x)\,y(x) = 0 \longrightarrow (1)$$

$$z''(x) + v(x)\,z(x) = 0 \longrightarrow (2)$$

and $y(x)$ and $z(x)$ are in non trivial soln.

$x_1, x_2, ..$ be 2 successive zero of z

$z(x_1) = 0$ & $z(x_2) = 0$.

$z(x)$ does not vanish in (x_1, x_2) and derive at a Contradiction.

Without loss of generality, we assume that $y(x)$ & $z(x)$ are +ve on (x_1, x_2)

If necessarily either their function would be replaced by their (-ve)

If we distinct the wronkskain.

By wronkskain we have

$$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$w(y, z) = \begin{vmatrix} y(x) & z(x) \\ y'(x) & z'(x) \end{vmatrix}$$

$w(y,z) = y(x)z'(x) - z(x)y'(x)$ is a function of x. Then we can write

$$w(x) = y(x)z'(x) - z(x)y'(x)$$

$$w(x) = yz' - zy' \longrightarrow (3)$$

differentiate w.r. To x.

$$\frac{dw(x)}{dx} = yz'' + y'z' - zy'' - y'z'$$

$$w' = yz'' - zy'' \longrightarrow (4)$$

we have

(1) $\rightarrow$ $y'' + qy = 0$ $\quad y'' = -qy$

(2) $\rightarrow$ $z'' + rz = 0$ $\quad z'' = -rz$

Illly $\frac{dw}{dx}(x) = y(-rz) - z(q/y)$

$$= -ry(z) + qz(y)$$

$\frac{dv}{dx}(x) > 0$ on (x_1, x_2).

Taking integrate on both sides x_1 to x_2 we have

$$\int_{x_1}^{x_2} \frac{dv}{dx}(x), dx > 0 \quad ; \quad \int_{x_1}^{x_2} dw(x) > 0$$

$w(x_2) - w(x_1) > 0$

$w(x_2) > w(x_1)$

(3) we have $w(x) = yz' - zy'$

$z(x_1) = 0$ & $z(x_2) = 0$

$w(x) = yz'$ on (x_1, x_2)

Then the wronkskain

$w(x) = z'(x)y(x)$ at (x_1, x_2).

||ly

$w(x_1) > 0$ $\quad$ $z(x_1) = 0$

$w(x_2) \leq 0$ $\quad$ $z(x_2) = 0$

which implies to our assumption

Hence $y(x)$ vanishes atleast one b/w any two successive zero's of z.

Eigen values and Eigen function and the vibration string.

Eigen values and eigen functions

A non trivial soln of $y(x)$ of eqn

$y'' + \lambda y = 0 \longrightarrow (1)$

The satisfied boundary condition

$y(0) = 0 \quad , \quad y(\pi) = 0 \longrightarrow (2)$

if λ is (-ve) then strum seperation thearem only the trivial soln of (1) can be satisfied (2)

if $\lambda = 0$ then general soln is,

$y(x) = C_1 \sin \pi x + C_2 \cos \pi x.$

If $x = 0$, $y = 0$.

if $x = \pi$, $y(\pi) = C \sin \sqrt{\lambda \pi} \longrightarrow (3)$

Thus the equation (3) satisfied the condition $y(\pi) = 0$. It is clear that $\sqrt{\pi x}$ must equal to $n\pi$. [non (-ve)] so

$\lambda = n^2$ in there words must equal If the number c, H, q

These values of A are called eigen values of the probability & corresponding soln $\sin x$, $\sin 2x$, $\sin 3x$ .... are called eigen functions.

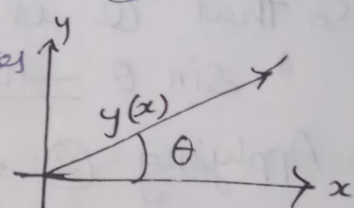
Vibration string

Suppose that a flexible pulled on the x axis and for tends to two paints for convenience Take $x=0$, & $x=\pi$ as starting & ending points of string.

The string is drawn a side into a certain curve $y=f(x)$ in the xy plane & released the equation of motion will make general simplifying the first of which is subsequence vibration is entirely traverse.

This means that each point of the x-co-ordinates string has constant

So that its y-coordinates depend only on x the time.



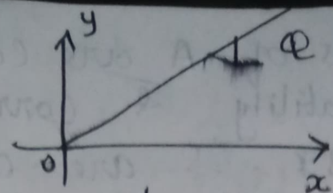
Explain one dimensional wave eqn.

This displace of the string from its equilibrium position is q_x by some function $y = y(x,t)$.

$y = y(x,t)$ the time derivatives $\frac{dy}{dt}$ & $\frac{d^2y}{dt^2}$ represent the string velocity & acceleration of string

Consider the motion of small piece that is equilibrium position has length ∂x.

If linear mass density of string is $m=n$ such that | mass of piece which is $m-\Delta x$ that by $F=ma$



Newton's second law of motion

The transverse force facting on it given by,

$$F = m \Delta x \frac{\partial^2 y}{\partial t^2} \longrightarrow (1)$$

The string is flexible $T = T(x)$ at any point is directed along the tangent and has $T \cdot \sin\theta$ as it y Component.

F is differente between the value Δ of T.

Since and the end of our piece namely,

$$\Delta(T \sin\theta) = m \Delta x \cdot \frac{\partial^2 y}{\partial t^2} \longrightarrow (2)$$

if the vibrations are relatively small So that θ is small &

$$\sin\theta \simeq \tan\theta = \frac{\partial y}{\partial x}$$

Applying (2) $\Rightarrow \Delta T\left(\frac{\partial y}{\partial x}\right) = m \frac{\partial^2 y}{\partial t^2}$

$$\frac{\partial y}{\partial x} T\left(\frac{\partial y}{\partial x}\right) = m \frac{\partial^2 y}{\partial t^2}$$

Here is m and T are Constants So that equation Can be written as

$$a^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \longrightarrow (3)$$

$$a = \sqrt{\frac{T}{m}}$$

Hence (3) is Called the 1-dimensional wave equation.

Here $y(x, t)$ satisfies the boundary Condition

$$y(0, t) = 0 \longrightarrow (4)$$
$$y(\pi, t) = 0 \longrightarrow (5)$$

Add the initial Conditions

$\left|\frac{\partial y}{\partial t}\right| t = 0 \longrightarrow (6)$

$y(x,0) = f(x) \longrightarrow (7)$

$y(x,t) = u(x) v(t) \longrightarrow (8)$

$(3) \Rightarrow a^2 u''(x) v(t) = u(x) v''(t)$

$\div$ by $u(x)$

$$\frac{u''(x)}{u(x)} = \frac{1}{a^2} \frac{v''(t)}{v(t)} \longrightarrow (A)$$

Since the left sides of functions only of x and the right side of the func only on t. The above eqn can be hold. only it both sides are Constant.

if we denote its Constant, then A

$y'' + \lambda y = 0$

$y'' + \lambda a^2 y = 0$

Splits into two O.D.E for $u(x)$ & $v(x)$

$u'' + \lambda u = 0 \longrightarrow (9)$

$v'' + \lambda a^2 v = 0 \longrightarrow (10)$

Since, $u(0) = u(\pi) = 0$ iff $\lambda = n^2$ for some +ve integer n the Corresponding soln are

$u_n(x) = \sin(nx)$

$v(t) = C_1 \sin(n\alpha t) + C_2 \cos(n \alpha t)$

$v_n t = \cos(n \alpha t)$

The product of soln is $y_n(x,t)$ $\sin nt + \cos nt$

in each of these func for $n = 1, 2, \ldots$

$b_1 \sin x \, 0 \, at + b_2 \sin 2n \cos 2at + \ldots + b_n \sin nx \cos nx$

If proceed term by differentiability Then any infinite series of form

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin x \cos nt$$

$$y(x,t) = b_1 \sin x \cos at + b_2 \sin x \cos 2at + \ldots + b_n \sin x \cos nat$$

is soln. satisfies $y(a,t) = 0$ $y(\pi, t) = 0$

initial shape of string.

$f(x) = b_1 \sin x + b_2 \sin 2x + \ldots + b_n \sin nx$ if $t = 0$ //.