UNIT-V

Taylor series method for simultaneous first order differential equation:

The simultaneous first order differential equation of the type $\frac{dy}{dx} = f(x,y,z)$, $\frac{dz}{dx} = g(x,y,z)$ with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ can be solved by using Taylor method. The method is explained in the following example.

$$\boxed{y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \cdots}$$

Example: 4 Using Taylor series method find approximate values of y and z corresponding to $x = 0.1, 0.2$ given that $y(0) = 2$ ($y(x_0)=y_0$), $z(0) = 1$ ($z(x_0)=z_0$) and

$$\frac{dy}{dx} = x + z, \quad \frac{dz}{dx} = x - y^2$$

$y_1 = y_0 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1'''$

Sol:

Given

$y' = x + z$, $z' = x - y^2$ and $x_0 = 0$, $y_0 = 2$, $z_0 = 1$, $h = 0.1$

$y' = x + z \qquad z' = x - y^2 \qquad uv = uv' + vu'$

$y'' = 1 + z' \qquad z'' = 1 - 2yy'$

$y''' = z'' \qquad z''' = -2[yy'' + y'^2]$

To find $y(0.1)$ and $z(0.1)$

Taylor series for y_1 is

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots \longrightarrow (1)$$

$$y_0' = (x+z)_0 = x_0 + z_0 = 1 \longrightarrow (2)$$

$$y_0'' = (1+z')_0 = 1 + z_0' = 1 + x_0 - y_0^2 = -3 \longrightarrow (3)$$

$$y_0''' = z_0'' = 1 - 2y_0y_0' = 1 - 4 = -3 \longrightarrow (4)$$

Substituting (2), (3) and (4) in (1) we get

$$y_1 = 2 + (0.1)1 + \frac{(0.1)^2}{2!}(-3) + \frac{(0.1)^3}{3!}(-3)$$

$$= 2.1 - 0.015 - 0.0005 = 2.0845$$

$y(x_1) = y_1$

$$\boxed{y(0.1) = 2.0845} \longrightarrow (5)$$

Taylor series for z_1 is

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} z_0''' + \cdots \longrightarrow (6)$$

$$z_0' = (x - y^2)_0 = (x_0 - y_0^2) = -4 \longrightarrow (7)$$

$$z_0'' = (1 - 2yy')_0 = 1 - 2y_0y_0' = 1 - 2\times2\times1 = -3 \longrightarrow (8)$$

$$z_0''' = -2[yy'' + y'^2]_0$$

$$= -2[y_0y_0'' + y_0'^2] = -2[2(-3)+1] = 10 \longrightarrow (9)$$

Substituting (7), (8) and (9) in (6) we get

$$z_1 = 1 + (0.1)(-4) + \frac{(0.1)^2}{2!}(-3) + \frac{(0.1)^3}{3!}(10)$$

$$= 1 - 0.4 - 0.015 + 0.001666 = 0.5866$$

$z(x_1) = z_1$

$$\boxed{z(0.1) = 0.5867} \longrightarrow (10)$$

To find $y(0.2)$ and $z(0.2)$

Taylor series for y_2 is

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots \longrightarrow (11)$$

$$y_1' = (x+z)_1 = x_1 + z_1 = 0.1 + 0.5867 = 0.6867 \longrightarrow$$

$$y_1'' = (1+z')_1 = (1 + x - y^2)_1$$

$$= 1 + x_1 - y_1^2 = 1 + 0.1 - (2.0845)^2 = -3.245 \longrightarrow$$

$$y_1''' = (z'')_1 = (1 - 2yy')_1$$

$$= 1 - 2(2.0845)(0.6867) = -1.8629 \longrightarrow$$

Substituting (12), (13) and (14) in (11) we get

$$y_2 = 2.0845 + (0.1)(0.6867) + \frac{(0.1)^2}{2!}(-3.245) + \frac{(0.1)^3}{3!}(-1.8628)$$

$$= 2.0845 + 0.06867 - 0.0162 - 0.00031$$

$$\boxed{y(0.2) = 2.1367} \longrightarrow (15)$$

Taylor series for z_2 is

$$z_2 = z_1 + hz_1' + \frac{h^2}{2!}z_1'' + \frac{h^3}{3!}z_1''' + \dots \longrightarrow (16)$$

$h = x_1 - x_0$

$= 0.1 - 0$

$h = 0.1$

$$z_1' = (x - y^2)_1 = (x_1 - y_1^2)$$

$$= 0.1 - 4.3451 = -4.2451 \longrightarrow (17)$$

$$z_1'' = (1 - 2yy')_1 = 1 - 2y_1y_1'$$

$$= 1 - 2(2.0845)(0.6867) = -1.8628 \longrightarrow (18)$$

$$z_1''' = -2[yy'' + y'^2]_1 = -2[y_1y_1'' + y_1'^2]$$

$$= -2[(2.0845)(-3.245) + (0.6867)^2]$$

$$= -2[-6.7642 + 0.4716] = 12.5852 \longrightarrow (19)$$

Substituting (17), (18), and (19) in (16) we get

$$z_2 = 0.5867 + (0.1)(-4.2451) + \frac{(0.1)^2}{2!}(-1.8629) + \frac{(0.1)^3}{3!}(12.5852)$$

1.167196

$$= 0.5866 - 0.4245 - 0.009315 + 0.00290$$

$$= 0.15497$$

$$\boxed{z(0.2) = 0.15497}$$

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using Taylor seri $y' = x + y$ $y(0) = 1$ $h = 0.1$

find out $y(0.1)$

Ans: 1.1103

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problem: 2 Given $\frac{dy}{dx} = 3x + \frac{y}{2}$ and $y(0) = 1$ find the values of $y(0.1)$ and $y(0.2)$ using Taylor series method.

Sol:

Given, $y' = 3x + \frac{y}{2}$; $x_0 = 0$; $y_0 = 1$

$$y_1(x) = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv}$$

$$y' = 3x + \frac{y}{2} \Rightarrow y_0' = 3x_0 + \frac{y_0}{2} = \frac{1}{2} = 0.5$$

$$y'' = 3 + \frac{y'}{2} \Rightarrow y_0'' = 3 + \frac{y_0'}{2} = 3 + \frac{(0.5)}{2} = 3.25$$

$$y''' = \frac{y''}{2} = y_0''' = \frac{y_0''}{2} = \frac{3.25}{2} = 1.625$$

$$y^{iv} = \frac{y'''}{2} = y_0^{iv} = \frac{y_0'''}{2} = \frac{1.625}{2} = 0.8125$$

$$y(0.1) = 1 + \frac{0.1}{1!}(0.5) + \frac{(0.1)^2}{2!}(3.25) + \frac{(0.1)^3}{3!}(1.625) + \frac{(0.1)^4}{4!}(0.8125)$$

$$= 1 + 0.05 + 0.01625 + 2.70834 + 3.38541$$

$$\boxed{\begin{array}{c} y(x_1) = y_1 \\ y(0.1) = 1.0665 \end{array}}$$

Taylor series for y_2 is

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1''''$$

$$y_1' = 3x_1 + \frac{y_1}{2} = 3(0.1) + \frac{1.0665}{2} = 0.8332$$

$$y_1'' = 3 + \frac{y_1'}{2} = 3 + \frac{0.8332}{2} = 3.4166$$

$$y_1''' = \frac{y_1''}{2} = \frac{3.4166}{2} = 1.7083$$

$$y_1'''' = \frac{y_1'''}{2} = \frac{1.7083}{2} = 0.8541$$

$$y(0.2) = 1.0665 + (0.1)(0.8332) + \frac{(0.1)^2}{2}(3.4166) + \frac{(0.1)^3}{6}(1.7083) + \frac{(0.1)}{24}(0.8541)$$

$$= 1.0665 + 0.0833 + 0.0171 + 0.0002 + 0.000003$$

$$\boxed{y(0.2) = 1.167103}$$

Problem: 3 Given $\frac{dy}{dx} = x + y$ and $y(0) = 1$. find the values of $y(0.1)$ using Taylor series method.

Sol:

Given, $\frac{dy}{dx} = x + y$; $x_0 = 0$; $y_0 = 1$; $h = 0.1$

$y' = x + y$

$y'' = 1 + y'$

$y''' = y''$

$y' = x_0 + y_0 = 0 + 1 = 1$

$y'' = 1 + 1 = 2$

$y''' = 2$

Taylor series for y_1 is

$$y(0.1) = y_0 + hy_1' + \frac{h^2}{2} y_1'' + \frac{h^3}{6} y_1'''$$

$$= 1 + (0.1)(1) + \frac{(0.1)^2}{2}(2) + \frac{(0.1)^3}{6}(2)$$

$$= 1 + 0.1 + 0.01 + 0.0003$$

$$\boxed{y(0.1) = 1.1103}$$

$$y_{m+1} = y_m + \frac{h}{2}\{f(x_m, y_m) + f[x_m + h, y_m + hf(x_m, y_m)]\}$$

Euler's Method (or) Improded:

$$y_1 = y_0 + \frac{h}{2}\{f(x_0, y_0) + f[x_0 + h, y_0 + hf(x_0, y_0)]\}$$

In general

$$y_{m+1} = y_m + \frac{h}{2}\{f(x_m, y_m) + f[x_m + h, y_m + hf(x_m, y_m)]\}$$

Modified Euler's Method:

$$y_1 = y_0 + hf\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right]$$

$$y_{n+1} = y_n + hf\left[x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right]$$

(or)

$$y(x+h) = y(x) + hf\left[x + \frac{h}{2}, y + \frac{h}{2} f(x, y)\right]$$

Problem: 1

Using improded Euler's method find y at $x = 0.1$ and $x = 0.2$ the given $\frac{dy}{dx} = y - \frac{2x}{y}$, $y(0) = 1$

Sol: $y_{m+1} = y_m + \frac{h}{2}\{f(x_m, y_m) + f[x_m + h, y_m + hf(x_m, y_m)]\}$

The imporded Euler's algorithm is

$$y_{m+1} = y_m + \frac{h}{2}\{f(x_m, y_m) + f[x_m + h, y_m + hf(x_m, y_m)]\} \quad (1)$$

put $m = 0$,

$$y_1 = y_0 + \frac{h}{2}\{f(x_0, y_0) + f[x_0 + h, y_0 + hf(x_0, y_0)]\} \rightarrow (2)$$

Here, $x_0 = 0$, $y_0 = 1$, $h = 0.1$, $f(x, y) = y - \frac{2x}{y}$

$$f(x_0, y_0) = y_0 - \frac{2x_0}{y_0} = 1 - \frac{2(0)}{1} = 1 \rightarrow (3)$$

Sub ③ in ②

$y_1 = 1 + \frac{0.1}{2}\{1 + f(0+0.1, 0.1(1))]$

$= 1 + \frac{0.1}{2}\{1 + f(0.1, 1.1)]$ ⟶ ⑤

$f(0.1, 1.1) = 1.1 - \frac{2(0.1)}{1.1} = 0.9182$ ⟶ ⑥

Sub ⑥ in ⑤ we get

$y_1 = 1 + \frac{0.1}{2}\{1 + 0.9182\}$

$= 1 + \frac{0.1}{2}\{1.9182\}$

$\boxed{y(0.1) = 1.09591}$

$y_{m+1} = y_m + \frac{h}{2}\{f(x_m, y_m) + f[x_m + h, y_m + h f(x_m, y_m)]$

put $m = 1$,

$y_2 = y_1 + \frac{h}{2}\{f(x_1, y_1) + f[x_1 + h, y_1 + h f(x_1, y_1)]$ ⟶ ⑥

Here, $x_1 = 0.1$, $y_1 = 1.09591$, $h = 0.1$, $f(x, y) = y - \frac{2x}{y}$

$f(x_1, y_1) = y_1 - \frac{2x_1}{y_1}$

$= 1.09591 - \frac{2(0.1)}{1.09591}$

$f(x_1, y_1) = 0.9134$ ⟶ ⑦

Sub ⑦ in ⑥ we get

$y_2 = 1.09591 + \frac{0.1}{2}\{0.9134 + f[0.1 + 0.1, 1.09591 + (0.1)(0.9134)]$

$y_2 = 1.09591 + \frac{0.1}{2}\{0.9134 + f(0.2, 1.1872)\}$ ⑧

Now,

$f(0.2, 1.1872) = 1.1872 - \frac{2(0.2)}{1.1872}$ ⟶ ⑨

$f(0.2, 1.1872) = 0.8503$

Sub ⑨ in ⑧ we get,

$y_2 = 1.09591 + \frac{0.1}{2}\{0.9134 + 0.8503\}$

$\boxed{y(0.2) = 1.1841}$

x	0	0.1	0.2
y	1	1.0959	1.1841

Modified Euler's Method:

Problem: 1

Given $\frac{dy}{dx} + y - x^2 = 0$, $y(0) = 1$, $y(0.1) = 0.9052$, $y(0.2) = 0.8213$ find correct to four decimal places $y(0.3)$ using modified method.

Sol: $y_{n+1} = y_n + h f\{x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\}$

The modified Euler's formula is

$y_{n+1} = y_n + h f[x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)]$

Given,

$x_0 = 0$, $y_0 = 1$, $\frac{dy}{dx} = x^2 - y$

$x_1 = 0.1$, $y_1 = 0.9052$, $f(x, y) = x^2 - y$

$x_2 = 0.2$, $y_2 = 0.8213$, $h = 0.1$

$x_3 = 0.3$; $y_3 = ?$

Put $n=2$ in (1)

$$y_3 = y_2 + 0.1 f\left[x_2 + \frac{0.1}{2}, y_2 + \frac{0.1}{2} f(x_2, y_2)\right]$$

$$y_3 = 0.8213 + 0.1 f\left[0.2 + \frac{0.1}{2}, 0.8213 + \frac{0.1}{2} f(0.2, 0.8213)\right]$$ (2)

Now,

$$f(0.2, 0.8213) = (0.2)^2 - 0.8213$$

$$= -0.7813 \longrightarrow (3)$$

Sub (3) in (2) we get

$$y_3 = 0.8213 + 0.1 f\left[0.25, 0.8213 + \frac{0.1}{2}(-0.7813)\right]$$

$$= 0.8213 + 0.1 f[0.25, 0.7822] \longrightarrow (4)$$

Now,

$$f(0.25, 0.7822) = (0.25)^2 - 0.7822$$

$$= -0.7197 \longrightarrow (5)$$

Sub (5) in (4) we get

$$y_3 = 0.8213 + 0.1(-0.7197)$$

$$\boxed{y(0.3) = 0.74933}$$

THE RUNGE - KUTTA METHODS

Second order runge - kutta method:

If the initial values are x_0, y_0 for the differential equation $\frac{dy}{dx} = f(x,y)$ then the first increment in y viz Δy is computed from the formulae.

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf(x_0 + h/2, y_0 + k_1/2)$$

$$\Delta y = k_2$$

Now, $x_1 = x_0 + h$, $y_1 = y_0 + \Delta y$

The increment in y for the second interval is computed in a similar manner by means of the formulae.

$$k_1 = hf(x_1, y_1)$$

$$k_2 = hf(x_1 + h/2, y_1 + k_1/2)$$

$$\Delta y = k_2$$

and so on for the succeding intervals

Fourth order runge - kutta method:

This method is mostly commonly used in practice. Unless and otherwise stated Runge kutta method means only fourth order runge kutta method. Let $\frac{dy}{dx} = f(x,y)$ be a differential equation to be solved under the condition $y(x_0) = y_0$. If 'h' be the length of the interval between equidistant value then the first increment in y is computed from the formulae

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf(x_0 + h/2, y_0 + k_1/2)$$

$$k_3 = hf(x_0 + h/2, y_0 + k_2/2)$$

$K_4 = hf(x_0+h, y_0+K_3)$

$\Delta y = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

Now,

$x_1 = x_0 + h, \quad y_1 = y_0 + \Delta y$

Problem: 2

Using Runge Kutta method of fourth order. Solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2, 0.4$

Sol:

Given, $y' = \frac{y^2 - x^2}{y^2 + x^2}$

Also we are given that $x_0 = 0, y_0 = 1, h = 0.2$

To find $y(0.2)$

$K_1 = hf(x_0, y_0)$

$K_1 = h\left(\frac{y_0^2 - x_0^2}{y_0^2 + x_0^2}\right) = (0.2)\left(\frac{1-0}{1+0}\right) = 0.2$

$K_2 = hf(x_0 + h/2, y_0 + K_1/2)$

$$= (0.2)\frac{\left(1+\frac{0.2}{2}\right)^2 - \left(0+\frac{0.2}{2}\right)^2}{\left(1+\frac{0.2}{2}\right)^2 + \left(0+\frac{0.2}{2}\right)^2}$$

$$K_2 = 0.2\left[\frac{1.21 - 0.01}{1.21 + 0.01}\right]$$

$$\boxed{K_2 = 0.19672}$$

$K_3 = hf(x_0 + h/2, y_0 + K_2/2)$

$$= \frac{0.2\left(1+\frac{0.19672}{2}\right)^2 - \left(0+\frac{0.2}{2}\right)^2}{\left(1+\frac{0.19672}{2}\right)^2 + \left(0+\frac{0.2}{2}\right)^2}$$

$$= 0.2\left[\frac{1.2064 - 0.01}{1.2064 + 0.01}\right]$$

$$= 0.2\left[\frac{1.1964}{1.2164}\right]$$

$$\boxed{K_3 = 0.1967}$$

$K_4 = hf(x_0 + h, y_0 + K_3)$

$$= (0.2)\left[\frac{(1+0.1967)^2 - (0+0.2)^2}{(1+0.1967)^2 + (0.0+2)^2}\right]$$

$$= 0.2\left[\frac{1.4320 - 0.04}{1.4320 + 0.04}\right]$$

$$\boxed{K_4 = 0.1891}$$

$\therefore \Delta y = \frac{1}{6}[K_1 + 2K_2 + 3K_3 + K_4]$

$= \frac{1}{6}[0.2 + 2(0.19672) + 2(0.1967) + 0.1891]$

$= \frac{1}{6}[0.2 + 0.3934 + 0.3934 + 0.1891]$

$\Delta y = 0.19598$

$y_1 = y_0 + \Delta y$

$= 1 + 0.19598$

$$\boxed{y_1 = 1.19598}$$

To find $y(0.4)$. Here, $x_1 = 0.2$, $y_1 = 1.1959$, $h = 0.2$

$$K_1 = hf(x_1, y_1)$$
$$= h\left(\frac{y_1^2 - x_1^2}{y_1^2 + x_1^2}\right)$$
$$= 0.2\left(\frac{1.1959^2 - 0.2^2}{1.1959^2 + 0.2^2}\right)$$
$$= 0.2\left[\frac{1.4301 - 0.04}{1.4301 + 0.04}\right]$$

$$\boxed{K_1 = 0.1891}$$

$$K_2 = hf(x_0 + h/2, y_0 + K_1/2)$$
$$= (0.2)\frac{\left(1.1959 + \frac{0.2}{2}\right) - \left(0.2 + \frac{0.2}{2}\right)}{\left(1.1959 + \frac{0.2}{2}\right) + \left(0.2 + \frac{0.2}{2}\right)}$$
$$= 0.2\left[\frac{1.6851 - 0.09}{1.6851 + 0.09}\right]$$

$$\boxed{K_2 = 0.1794}$$

$$K_3 = hf(x_1 + h/2, y_1 + K_2/2)$$
$$K_3 = 0.2\left[\frac{\left(1.1959 + \frac{0.17948}{2}\right)^2 - \left(0.2 + \frac{0.2}{2}\right)^2}{\left(1.1959 + \frac{0.17948}{2}\right)^2 + \left(0.2 + \frac{0.2}{2}\right)^2}\right]$$
$$= 0.2\left[\frac{1.6529 - 0.09}{1.6929 + 0.09}\right]$$

$$\boxed{K_3 = 0.1793}$$

$$K_4 = hf(x_1 + h, y_1 + K_3)$$
$$= 0.2\left[(1.1959 + 0.1793)^2 - (0.2 + 0.2)^2\right]$$
$$= 0.2(1.8911 - 0.16)$$

$$\boxed{K_4 = 0.1687}$$

problem:1 By Applying the fourth order Runge-Kutta method find $y(0.2)$ from $y' = y - x$, $y(0) = 2$, taking $h = 0.1$

Sol:

Given, $y' = y - x$ (or) $f(x,y) = y - x$ and $y(0) = 2$,

$x_0 = 0$, $y_0 = 2$, $h = 0.1$

The fourth order Runge-Kutta method

$$\Delta y = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

Where,

$$K_1 = hf(x_0, y_0)$$
$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$
$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$
$$K_4 = hf(x_0 + h, y_0 + K_3)$$

$$K_1 = hf(x_0, y_0)$$
$$= 0.1(y_0 - x_0)$$
$$= 0.1 \times 2$$

$$\boxed{K_1 = 0.2}$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$
$$= 0.1\left[y_0 + \frac{K_1}{2} - \left(x_0 + \frac{h}{2}\right)\right]$$

$$= 0.1\left[2 + \frac{0.2}{2} - \left(\frac{0.1}{2}\right)\right]$$

$$= 0.1\,[2.1 - 0.05]$$

$$\boxed{K_2 = 0.205}$$

$$K_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_2}{2}\right)$$

$$= 0.1\left[y_0 + \frac{K_2}{2} - x_0 + \frac{h}{2}\right]$$

$$= 0.1\left[2 + \frac{0.205}{2} - \left(\frac{0.1}{2}\right)\right]$$

$$= 0.1\,[2.1025 - 0.05]$$

$$\boxed{K_3 = 0.20525}$$

$$K_4 = hf(x_0 + h,\ y_0 + K_3)$$

$$= 0.1\,[y_0 + K_3 - (x_0 + h)]$$

$$= 0.1\,[2 + 0.20525 - 0.1]$$

$$= 0.1\,[2.20525 - 0.1]$$

$$\boxed{K_4 = 0.210525}$$

$$\Delta y = \frac{1}{6}\,(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= \frac{1}{6}\left(0.2 + 2\,(0.205) + 2\,(0.20525) + 0.21052\right)$$

$$\boxed{\Delta y = 0.20517}$$

$$y(0.1) = y_0 + \Delta y$$

$$= 2 + 0.20517$$

$$\boxed{y_1 = 2.20517}$$

Next we go to find $y(0.2)$: $y_2 = y_1 + \Delta y$

where

$$\Delta y = \frac{1}{6}\,(K_1 + 2K_2 + 2K_3 + K_4)$$

where, $x_1 = x_0 + h$

$$= 0 + 0.1$$

$$\boxed{x_1 = 0.1}$$

Now,

$$K_1 = hf(x_1, y_1)$$

$$= 0.1\,(y_1 - x_1)$$

$$= 0.1\,(2.10517)$$

$$\boxed{K_1 = 0.2105}$$

$$K_2 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{K_1}{2}\right)$$

$$= 0.1\left(y_1 + \frac{K_1}{2} - \left(x_1 + \frac{h}{2}\right)\right)$$

$$= 0.1\left[2.20517 + \frac{0.2105}{2} - \left(0.1 + \frac{0.1}{2}\right)\right]$$

$$= 0.1\,[2.31042 - 0.15]$$

$$\boxed{K_2 = 0.216042}$$