UNIT - V

FOURIER SERIES

Fourier Series

Definition:

If $f(x)$ is a periodic function and satisfies Dirichlet conditions (to be described in subsequent article), then it can be represented by an infinite series called Fourier series as,

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots + a_n \cos nx + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots + b_n \sin nx + \cdots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- (1)}$$

Where a_0, a_n & b_n are called fourier coefficients.

Dirichlet conditions

Suppose that.

(i) $f(x)$ is defined and single valued except possibly at a finite number of points in $(-\pi, \pi)$

(ii) $f(x)$ is periodic with period 2π.

(iii) $f(x)$ & $f'(x)$ are piecewise continuous in $(-\pi, \pi)$. Then the above series (1) converges to

(a) $f(x)$ if x is a point of continuity

(b) $\frac{f(x+0) + f(x-0)}{2}$ if x is a point of discontinuity

Therefore the value of $f(x)$ at any point of continuity x in $(-\pi, \pi)$ is given by $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$.

The value of $f(x)$ at any point of continuity x in $(-\pi, \pi)$ is given by

$$\frac{f(x+0) + f(x-0)}{2}$$

Note 1: Let the function $f(x) = x^2$ be defined in the interval $-\pi < x < \pi$. Let its fourier series be,

$$f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \quad \text{--(1)}$$

At $x=0$ ('0' lies within $-\pi < x < \pi$ and hence it is a point of continuity) the sum of the fourier series (1) equal to $f(0)$ (which is zero here).

i.e, $$f(0) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$0 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

Note 2:

suppose $f(x)$ is defined in the interval $-\pi < x < \pi$ then at the points of discontinuities (at $x = \pi$ or $x = -\pi$), the sum of the fourier series is equal to the arithmetic mean of the value of $f(x)$ at $x = \pi$ & $x = -\pi$

i.e, sum of the series $x = \pi$ is equal to

$$\frac{f(\pi) + f(-\pi)}{2}$$

Note 3

suppose the function $f(x)$ is defined by

$$f(x) = \begin{cases} -\pi & \text{in } -\pi < x < 0 \\ x & \text{in } 0 < x < \pi \end{cases},$$

then at the point of discontinuity $x=0$ (which is the middle of the given interval) the sum of the fourier series converges (or equals) to the average value of the right hand limit and left hand limit of the given function

(c) Sum of the series at $x=0$ is equal to

Here sum of the series $\frac{f(0+0)+f(0-0)}{2}$

$= \frac{0-\pi}{2} = -\pi/2$.

Note 4: If x is a point of continuity, then the sum of the fourier series is equal to $f(x)$

i.e., Sum of the series $= f(x)$.

Determine the fourier coefficients a_0, a_n & b_n

The fourier series for the function $f(x)$ in the interval $c < x < c+2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where,

$$\left.\begin{aligned} a_0 &= 1/\pi \int_c^{c+2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx \end{aligned}\right\} \quad -(1).$$

The values of a_0, a_n, b_n are known as Euler's Formulae. To establish these formulae, the following results will be required.

1. $\int_c^{c+2\pi} \cos nx\,dx = 0 \quad (n \neq 0)$

2. $\int_c^{c+2\pi} \sin nx\,dx = 0 \quad (n \neq 0)$

3. $\int_c^{c+2\pi} \cos mx \cos nx\,dx = 0 \quad (m \neq n)$.

4. $\int_c^{c+2\pi} \cos^2 nx\,dx = \pi \quad (n \neq 0)$.

5. $\int_{c}^{c+2\pi} \sin mx \cos nx \, dx = 0 \quad (m \neq n)$

6. $\int_{c}^{c+2\pi} \sin nx \cos nx \, dx = 0$

7. $\int_{c}^{c+2\pi} \sin mx \sin nx \, dx = 0 \quad (m \neq n)$

8. $\int_{c}^{c+2\pi} \sin^2 nx = \pi \quad (n \neq 0)$

<u>Proof</u> of <u>Euler</u> formulae:

Let $f(x)$ be represented in the interval $(c, c+2\pi)$ by the fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \to (1)$$

To find the coefficients a_0, a_n, b_n we assume that the series (1) can be integrated term by term from $x = c$ to $x = c + 2\pi$.

To find a_0, integrate both sides of (1) from $x = c$ to $x = c + 2\pi$. Then,

$$\int_{c}^{c+2\pi} f(x)\,dx = \frac{1}{2} a_0 \int_{c}^{c+2\pi} dx + \int_{c}^{c+2\pi} \left(\sum_{n=1}^{\infty} a_n \cos nx\right) + \int_{c}^{c+2\pi} \left(\sum_{n=1}^{\infty} b_n \sin nx\right) dx$$

$$= \frac{a_0}{2} (x)_{c}^{c+2\pi} + \sum_{n=1}^{\infty} a_n \int_{c}^{c+2\pi} \cos nx \, dx + \sum_{n=1}^{\infty} b_n \int_{c}^{c+2\pi} \sin nx \, dx.$$

$$= \frac{1}{2} a_0 \left[(c + 2\pi - c) + 0 + 0\right]$$

by (1) & (2)

$$= a_0 \pi.$$

$$\therefore a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x)\,dx.$$

To find a_n multiply each side of (1) by $\cos nx$ and integrate from $x = c$ to $x = c + 2\pi$. then,

$$\int_c^{c+2\pi} f(x) \cos nx \, dx = \frac{1}{2} a_0 \int_c^{c+2\pi} \cos nx \, dx + \int_c^{c+2\pi} \left(\sum_{n=1}^{\infty} a_n \cos nx \right) \cos nx \, dx + \int_c^{c+2\pi} \left(\sum_{n=1}^{\infty} b_n \sin nx \right) \cos nx \, dx$$

$$= \frac{1}{2} a_0 (0) + \sum_{n=1}^{\infty} a_n \int_c^{c+2\pi} \cos nx \cos nx \, dx + \sum_{n=1}^{\infty} b_n \int_c^{c+2\pi} \sin nx \cos nx \, dx$$

$$= 0 + \pi a_n + 0$$ (by integrals (1), (4) & (6))

$$\therefore a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx \, dx.$$

To find b_n, multiply each side of (1) by $\sin nx$ and integrate from $x = c$ to $x = c + 2\pi$, then,

$$\int_c^{c+2\pi} f(x) \sin nx \, dx = \frac{a_0}{2} \int_c^{c+2\pi} \sin nx \, dx + \int_c^{c+2\pi} \left(\sum_{n=1}^{\infty} a_n \cos nx \right) \sin nx \, dx + \int_c^{c+2\pi} \left(\sum_{n=1}^{\infty} b_n \sin nx \right) \sin nx \, dx$$

$$= 0 + 0 + \pi b_n$$ (by integrals (2), (6) & (8))

$$\therefore b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx \, dx.$$

Corollary (1): putting $c = 0$, the interval becomes $0 < x < 2\pi$, and the formula (1) reduces to

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$$

Write the formulae for fourier constants for $f(x)$ in the interval $(-\pi, \pi)$

Corollary (2): putting $c = -\pi$, the interval becomes $-\pi < x < \pi$, and the formula (1) becomes,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

Note: The following two results are very useful for this chapter and solving of boundary value problem

$\cos n\pi = (-1)^n$, where 'n' is an integer

$\sin n\pi = 0$, where 'n' is an integer

Fourier series in the interval $(-\pi, \pi)$

Ex: 1 Find the fourier series for

$$f(x) = \begin{cases} -k ; & -\pi < x < 0 \\ k , & 0 < x < \pi \end{cases}$$

Hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$

Soln:

Step 1: WKT, the fourier series of $f(x)$ in $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$$

Step 2: To find a_0

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

$$= 1/\pi\left[\int_{-\pi}^{0} -k\,dx + \int_{0}^{\pi} k\,dx\right]$$

$$= k/\pi\left[(-x)_{-\pi}^{0} + (x)_{0}^{\pi}\right]$$

$$= \frac{k}{\pi}\left[0 - (-\pi) + \pi\right]$$

$$\therefore \boxed{a_0 = 0} \rightarrow (2)$$

Step 3: To find a_n

$$a_n = 1/\pi \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$= 1/\pi\left[\int_{-\pi}^{0} -k \cos nx\,dx + \int_{0}^{\pi} k \cos nx\,dx\right]$$

$$= \frac{k}{\pi}\left[\left(-\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(\frac{\sin nx}{n}\right)_{0}^{\pi}\right]$$

$$= \frac{k}{\pi}\left[0 + 0\right]$$

$$\therefore \boxed{a_n = 0} \rightarrow (3)$$

Step 4: To find b_n

$$b_n = 1/\pi \int_{-\pi}^{\pi} f(x) \sin nx\,dx$$

$$= 1/\pi\left[\int_{-\pi}^{0} -k \sin nx\,dx + \int_{0}^{\pi} k \sin nx\,dx\right]$$

$$= k/\pi\left[\left(\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left(-\frac{\cos nx}{n}\right)_{0}^{\pi}\right]$$

$$= \frac{k}{\pi}\left[\frac{1}{n} - \frac{\cos n\pi}{n} - \frac{\cos n\pi}{n} + \frac{1}{n}\right]$$

$$= \frac{k}{\pi}\left[\frac{2}{n} - \frac{2\cos n\pi}{n}\right] = \frac{2k}{n\pi}\left[1-(-1)^n\right]$$

$b_n = 0$, n is even

$= \frac{4k}{n\pi}$, n is odd $\rightarrow (4)$

substituting (2), (3) & (4) in (1), we get

$$\therefore f(x) = \sum_{n=1,3,5}^{\infty} \frac{4k}{n\pi} \sin nx$$

i.e., $$f(x) = \frac{4k}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin(2n-1)x$$

<u>Ex: 2</u> Expand $f(x) = (\pi - x)^2$ in $(-\pi, \pi)$ as a fourier series

<u>Solu</u>:- WKT, a fourier series for the

<u>step 1</u>: function $f(x)$ in the interval $(-\pi, \pi)$ is given by,

$$f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos nx + \sum_1^{\infty} b_n \sin nx \rightarrow (A)$$

<u>step 2</u>: To find a_0

Here, $f(x) = (\pi - x)^2$ in $(-\pi, \pi)$

Now, $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - x)^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{(\pi - x)^3}{-3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[0 - \frac{(2\pi)^3}{-3}\right] = \frac{1}{\pi}\left[\frac{8\pi^3}{3}\right]$$

$$\therefore \boxed{a_0 = \frac{8\pi^2}{3}} \rightarrow (1)$$

<u>step 3</u>: To find a_n.

$$a_n = 1/\pi \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - x)^2 \cos nx \, dx$$

$u = (\pi - x)^2$	$v = \cos nx$
$u' = 2(\pi - x)(-1)$	$v_1 = \frac{\sin nx}{n}$
$u'' = 2$	$v_2 = \frac{-\cos nx}{n^2}$
$u''' = 0$	$v_3 = -\frac{\sin nx}{n^3}$

$$= \frac{1}{\pi}\left[(\pi - x)^2\left(\frac{\sin nx}{n}\right) - 2(\pi - x)(-1)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

(using Bernoulli's formula)

$$= \frac{1}{\pi}\left[(0+0+0) - \left(0 - 2(2\pi)(-1)\left(\frac{-\cos n\pi}{n^2}\right) + 0\right)\right]$$

$$= \frac{1}{\pi}\left[(0) - \left(\frac{-4\pi \cos n\pi}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{4\pi \cos n\pi}{n^2}\right]$$

$$\therefore a_n = \frac{4}{n^2}(-1)^n \quad \to (2)$$

<u>step 4</u>: To find b_n

$u = (\pi - x)^2$	$v = \sin nx$
$u' = 2(\pi - x)(-1)$	$v_1 = \frac{-\cos nx}{n}$
$u'' = 2$	$v_2 = \frac{-\sin nx}{n^2}$
$u''' = 0$	$v_3 = \frac{\cos nx}{n^3}$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - x)^2 \sin nx \, dx$$

$$= \frac{1}{\pi}\left[(\pi - x)^2\left(\frac{-\cos nx}{n}\right) - 2(\pi - x)(-1)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos n\pi}{n^3}\right)\right]_{-\pi}^{\pi}$$

(using Bernoulli's formula)

$$= \frac{1}{\pi}\left[\left(0 - 0 + \frac{2\cos n\pi}{n^3}\right) - \left(\frac{-(2\pi)^2 \cos nx}{n} - 0 + \frac{2\cos n\pi}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{2\cos n\pi}{n^3}\right) - \left(-\frac{4\pi^2 \cos n\pi}{n} + \frac{2\cos n\pi}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{4\pi^2 \cos n\pi}{n}\right] \quad \therefore b_n = \frac{4\pi}{n}(-1)^n \quad \to ($$

step 5: The required fourier series substituting (1), (2), (3) in (A), we get

$$f(x) = \frac{4\pi^2}{3} + \sum_1^{\infty} \frac{4(-1)^n}{n^2} \cos nx + \sum_1^{\infty} \frac{4\pi}{n} (-1)^n \sin nx$$

$$= \frac{4\pi^2}{3} + 4\sum_1^{\infty} \frac{(-1)^n}{n^2} \cos nx + 4\pi \sum_1^{\infty} \frac{(-1)^n \sin nx}{n}$$

Ex: 3 Find the fourier series for the function $f(x) = e^x$ defined in $(-\pi, \pi)$

Soln:-

step 1: WKT, a fourier series for the function $f(x)$ in $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (A)$$

step 2: To find a_0.

Here $f(x) = e^x$

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx$

$$= \frac{1}{\pi}\left[e^x\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[e^{\pi} - e^{-\pi}\right]$$

$$\therefore a_0 = \frac{2}{\pi} \sinh \pi \rightarrow (1)$$

$$\therefore \frac{e^x + e^{-x}}{2} = \sinh x.$$

step 3: To find a_n.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1^2 + n^2}(1 \cos nx + n \sin nx)\right]_{-\pi}^{\pi}$$

$$\left[\because \int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)\right]$$

$$= \frac{1}{(1^2+n^2)\pi}\left[e^{\pi}\cos n\pi - e^{-\pi}\cos n\pi\right]$$

$$= \frac{\cos n\pi\,(e^{\pi}-e^{-\pi})}{\pi(1+n^2)}$$

$$\therefore a_n = \frac{2(-1)^n \sinh\pi}{\pi(1+n^2)} \rightarrow (2)$$

Step 4: To find b_n.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{x}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{x}}{1+n^2}(\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$= \left\{\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right\}$$

$$= \frac{1}{\pi(1+n^2)}\left[-ne^{\pi}\cos n\pi + ne^{-\pi}\cos n\pi\right]$$

$$= \frac{n(-1)^n(e^{-\pi}-e^{\pi})}{\pi(1+n^2)}$$

$$\therefore b_n = \frac{2n(-1)^{n+1}\sinh\pi}{\pi(1+n^2)} \rightarrow (3)$$

Step 5: The required fourier series

Now we have,

$$a_0 = \frac{2\sinh\pi}{\pi}, \quad a_n = \frac{2(-1)^n\sinh\pi}{\pi(1+n^2)},$$

$$b_n = \frac{2n(-1)^{n+1}\sinh\pi}{\pi(1+n^2)}.$$

substituting these values in (A) we get,

$$f(x) = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\left[\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\cos nx - \sum_{n=1}^{\infty} n\frac{(-1)^n}{1+n^2}\sin nx\right]$$

$$= -\left(-\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots\right)$$

ie, $\frac{\pi - 2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} - \frac{1}{5\cdot 7} - \cdots$

Even and odd functions

A function $f(x)$ is said to be even if $f(-x) = f(x)$

Ex: x^2, $\cos x$, $\sin^2 x$, $|x|$, $x \sin x$ are even functions.

A function $f(x)$ is said to be odd if $f(-x) = -f(x)$

Ex: x^3, $\sin x$, $\tan^3 x$ are odd function

knowing that a function $f(x)$ is even or odd, can help us to avoid unnecessa work in computing the fourier coefficient of $f(x)$ which is based on the following facts.

Note 1:

odd × odd = even

even × even = even

odd × even = odd

even × odd = odd

Note 2: $\int_{-\pi}^{\pi} f(x)\,dx = 0$ if $f(x)$ is an odd function

(or)

$\int_{-\pi}^{\pi} f(x)\,dx = 2\int_{0}^{\pi} f(x)\,dx$, if $f(x)$ is an even function

similarly,

$\int_{-l}^{l} f(x)\,dx = 0$, if $f(x)$ is an odd fun

(or)

$\int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx$ if $f(x)$ is an even function.

Note 3: When $f(x)$ is an even function, the Euler's coefficients becomes

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

($\because$ Both $f(x)$ & $\cos nx$ are even, the product $f(x)\sin nx$ is odd function).

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

($\because$ $f(x)$ is even, $\sin nx$ is odd, the product $f(x)\sin nx$ is odd function).

$\therefore$ if a function $f(x)$ is even, its fourier expansion contains only cosine terms.

i.e., $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$, where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \quad ; \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

Note 4: If $f(x)$ is an odd function, then its fourier expansion contains only sine terms

i.e., $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$.

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

since $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$ ($\because$ $f(x)$ is odd)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$

Note 5: Even and odd functions cases can be used only when the function $f(x)$ is defined in $(-\pi, \pi)$ or $(-l, l)$.

Suppose $f(x) = x^2$ or $f(x) = x$ are defined in $(0, 2\pi)$ or $(0, 2l)$, we cannot use the above results

Fourier series of an odd function in the Interval $(-\pi, \pi)$

Ex: 1 prove that for $-\pi < x < \pi$,

$$\frac{x(\pi^2 - x^2)}{12} = \frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} - \frac{\sin 3x}{3^3} \dots$$

Soln:

Step 1: Let $f(x) = x(\pi^2 - x^2)$.

Now, $f(-x) = -x(\pi^2 - x^2)$

$= -[x(\pi^2 - x^2)] = -f(x)$

Hence the given function is an odd function

Therefore its fourier series given by.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \rightarrow (1).$$

Step 2: To find b_n.

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{x(\pi^2 - x^2)}{12} \sin nx \, dx$$

$u = x(\pi^2 - x^2)$ $\quad v = \sin nx$

$u' = \pi^2 - 3x^2$ $\quad v_1 = \frac{-\cos nx}{n}$

$u'' = -6x$ $\quad v_2 = -\frac{\sin nx}{n^2}$

$u''' = -6$ $\quad v_3 = \frac{\cos nx}{n^3}$

$u^{iv} = 0$ $\quad v_4 = \frac{\sin nx}{n^4}$

$$= \frac{2}{\pi} \cdot \frac{1}{12} \int_0^{\pi} x(\pi^2 - x^2) \sin nx \, dx$$

(odd function $\times$ odd function = even function)

$$= \frac{1}{6\pi} \left[(\pi^2 x - x^2) \left(\frac{-\cos nx}{n} \right) - (\pi^2 - 3x^2) \left(-\frac{\sin nx}{n^2} \right) + (-6x) \left(\frac{\cos nx}{n^3} \right) - (-6) \left(\frac{\sin nx}{n^4} \right) \right.$$

$= \frac{1}{6\pi}\left[\frac{-6\pi \cos n\pi}{n^3}\right]$ (using Bernoulli's formula).

$= \frac{-\cos n\pi}{n^3} = \frac{-(-1)^n}{n^3} = \frac{(-1)^{n+1}}{n^3} \rightarrow (2)$

Step 3: The required fourier series substituting (2) in (1), we get

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \sin nx$$

$$\therefore \frac{x(\pi^2 - x^2)}{12} = \frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3}$$

Ex: 2 If 'a' is neither zero nor an integer, find the fourier series expansion of period 2π for the function $f(x) = \sin ax$, in $-\pi \le x \le \pi$

Soln:-

Step 1: Here $f(x) = \sin ax$ is odd function. Hence the fourier coefficients $a_0 = 0, a_n = 0$. Therefore the fourier series for the function $f(x)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \, dx \rightarrow (A)$$

Step 2: To find b_n.

Now, $b_n = \frac{2}{\pi}\int_0^{\pi} \sin ax \sin nx \, dx$

$$= \frac{1}{\pi}\left[\int_0^{\pi} [\cos(n-a)x - \cos(n+a)x] \, dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(n-a)x}{(n-a)} - \frac{\sin(n+a)x}{(n+a)}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(n-a)\pi}{n-a} - \frac{\sin(n+a)\pi}{n+a}\right]$$

$(\because \sin 0 = 0)$

$$= \frac{1}{\pi}\left[\frac{\sin n\pi \cos a\pi - \cos n\pi \sin a\pi}{n-a} - \frac{\sin n\pi \cos a\pi}{n+a} - \frac{\cos n\pi \sin a\pi}{n+a}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n(-\sin a\pi)}{n-a} + \frac{(-1)^{n+1}\sin a\pi}{n+a}\right] \quad (\because \sin n\pi = 0)$$

(Note: that $\sin a\pi \neq 0$, since a is not an integer)

$$= \frac{(-1)^{n+1}\sin a\pi}{\pi}\left[\frac{1}{n-a} + \frac{1}{n+a}\right]$$

$$= (-1)^{n+1}\frac{2n \sin a\pi}{\pi(n^2-a^2)}$$

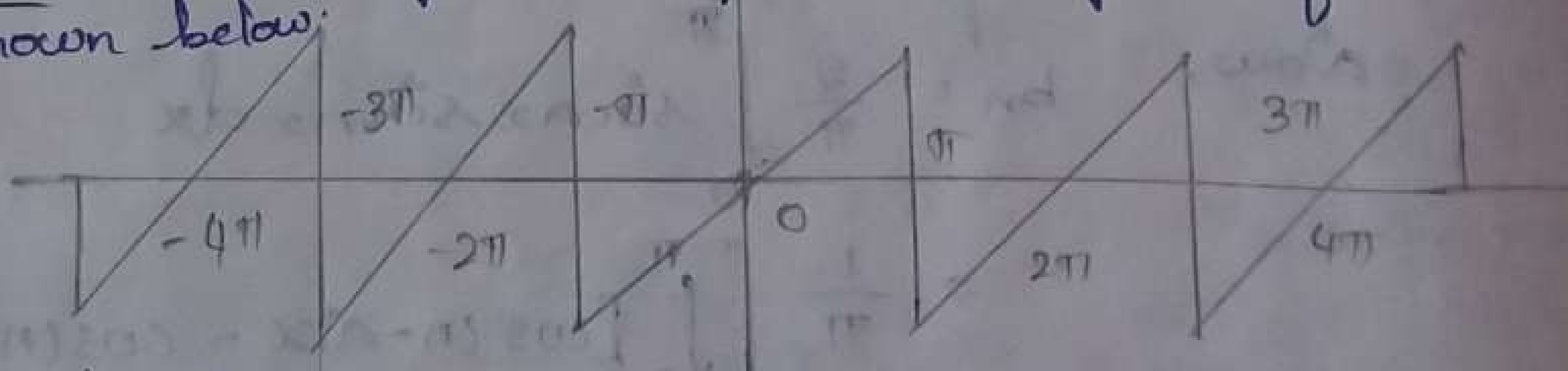
ie, $b_n = (-1)^{n+1}\dfrac{2n\sin a\pi}{\pi(n^2-a^2)} \rightarrow (B)$

Step 3: The required fourier series substituting (B) in (A) we get,

$$\therefore f(x) = \sin ax = \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{n^2-a^2}$$

Ex: 3 Show that the fourier series for $f(x) = x, -\pi < x < \pi$ is given by $f(x) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{\sin nx}{n}$.

Soln: The graph of y the given function shown below:



Step 1: Given $f(x) = x$

$f(-x) = -x$

ie, $f(x) = f(-x) = -f(x)$

$\therefore f(x) = x$ is an odd func

Hence $a_0 = 0,\ a_n = 0$

$\therefore$ The fourier series for the function

$f(x)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \to (1)$$

step 2: To find b_n

Now,

$u = x \qquad v = \sin nx$

$u' = 1 \qquad v_1 = \frac{-\cos nx}{n}$

$u'' = 0 \qquad v_2 = -\frac{\sin nx}{n^2}$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

(using Bernoulli's formula)

$$= \frac{1}{\pi}\left[\frac{-\pi \cos n\pi}{n} - (-\pi)\left(\frac{\cos n\pi}{n}\right)\right]$$

$$= \frac{1}{n}\left[-2\cos n\pi\right] = \frac{2}{n}(-1)^{n+1}$$

$$\text{ie., } \quad b_n = \frac{2}{n}(-1)^{n+1} \quad \to (2)$$

step 3: The required fourier series

substituting (2) in (1) we get,

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx$$

$$f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n}$$

$n=2 \quad 1-n^2$

Fourier series in the interval $(0, 2\pi)$

Ex: 1 Express $f(x) = x \sin x$ as a fourier series in $0 \le x \le 2\pi$.

Soln:

Step 1: WKT, a fourier series for the function $f(x)$ in the interval $[0, 2\pi]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (A)$$

Step 2: To find a_0

Here $f(x) = x \sin x$

Now,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x\,dx$$

$$= \frac{1}{\pi}\left\{x \cdot (-\cos x) - 1 \cdot (-\sin x)\right\}_0^{2\pi}$$

$$= \frac{1}{\pi}[-2\pi] = -2$$

$$\therefore \boxed{a_0 = -2} \rightarrow (1) \qquad (\because \sin 2\pi = 0, \ \cos 2\pi = 1)$$

Step 3: To find a_n

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$\left(\because \cos A \sin B = \tfrac{1}{2}\left\{\sin(A+B) - \sin(A-B)\right\}\right)$$

$$= \frac{1}{2\pi}\left[x \cdot \left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - \left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

(using Bernoulli's formula)

$$= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1}\right\}\right]$$

$(\because \sin 2(n+1)\pi = 0, \ \sin 2(n-1)\pi = 0$ &
$\cos 2(n+1)\pi = 1, \ \cos 2(n-1)\pi = 1$, whether n is odd or even).

$a_n = -\frac{1}{n+1} + \frac{1}{n-1} = \frac{2}{n^2-1}$ provided $n \neq 1$

$a_n = 2/n^2-1 \rightarrow (2)$

When $n=1$, we have

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x \, dx$$

$(\because \sin 2x = 2\sin x \cos x)$

$$= \frac{1}{2\pi}\left\{ x\left(-\frac{\cos 2x}{2}\right) - 1\left(\frac{\sin 2x}{4}\right)\right\}_0^{2\pi}$$

$$= \frac{1}{2\pi}[-\pi] = -1/2$$

$$\therefore \boxed{a_n = -1/2} \rightarrow (3)$$

Step 4: To find b_n

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\{\cos(n-1)x - \cos(n+1)x\} \, dx$$

$(\because 2\sin A \sin B = \cos(A-B) - \cos(A+B))$

$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right\} - 1\cdot\left\{-\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}$$

(using Bernoulli's formula).

$$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$\therefore \boxed{b_n = 0}$ provided $n \neq 1$ $\rightarrow (4)$

When $n=1$, we have

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x) \, dx$$

$u = x$ $v = 1-\cos$

$u' = 1$ $v_1 = \left(x - \frac{\sin}{}\right.$

$u'' = 0$ $v_2 = \left(\frac{x^2}{2} + \frac{\cos}{}\right.$

$$= \frac{1}{2\pi}\left\{x\left(x-\frac{\sin 2x}{2}\right) - 1\cdot\left(\frac{x^2}{2}+\frac{\cos 2x}{4}\right)\right\}_0^{2\pi}$$

(using Bernoulli's formula)

$$= \frac{1}{2\pi}\left[2\pi(2\pi) - \frac{4\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right]$$

$$\boxed{b_1 = \pi} \quad \rightarrow (5)$$

Step 5: The required Fourier series

From (A) we get,

$$f(x) = \frac{a_0}{2} + a_1\cos x + b_1 \sin x + \sum_{n=2}^{\infty} a_n \cos nx + \sum_{n=2}^{\infty} b_n \sin nx.$$

From (1), (2), (3), (4), & (5),

we get,

$a_0 = -2$, $a_n = \frac{2}{n^2-1}$, $(n \neq 1)$, $a_1 = -\frac{1}{2}$, $b_n = 0$, $b_1 = \pi$

substituting these values in (A) we get,

$$f(x) = -1 - \frac{1}{2}\cos x + \pi \sin x + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx$$

$$x \sin x = -1 + \pi \sin x - \frac{1}{2}\cos x + \frac{2}{2^2-1}\cos 2x + \frac{2}{3^2-1}\cos 3x + \cdots$$

Ex: 2 Express $f(x) = (\pi - x)^2$ as a fourier series of period 2π in the interval $0 < x < 2\pi$. Hence deduce the sum of the series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}$

Soln:

Step 1: WKT,

a fourier series for the function $f(x)$ in the interval $(0, 2\pi)$ is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad -(A)$$

Step 2: To find a_0

Here, $f(x) = (\pi - x)^2$ in $[0, 2\pi]$

Now, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi}(\pi - x)^2\,dx$

$$= \frac{1}{\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right] = \frac{2\pi^2}{3}$$

$$\therefore \boxed{a_0 = 2\pi^2/3} \quad \rightarrow (1)$$

Step 3: To find a_n

$u = (\pi - x)^2$ $\quad v = \cos nx$

$u' = 2(\pi - x)(-1)$ $\quad v_1 = \frac{\sin nx}{n}$

$u'' = 2$ $\quad v_2 = -\frac{\cos nx}{n^2}$

$u''' = 0$ $\quad v_3 = -\frac{\sin nx}{n^3}$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2 \cos nx\,dx$$

$$= \frac{1}{\pi}\left[(\pi - x)^2\left(\frac{\sin nx}{n}\right) - 2(\pi - x)(-1)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{\sin nx}{n^3}\right.\right.$$

(using Bernoulli's formula)

$$= \frac{1}{\pi}\left[\frac{2\pi\cos 2n\pi}{n^2} + \frac{2\pi}{n^2}\right] \quad (\because \cos 2n\pi = 1)$$

$$= 4/n^2$$

$$\boxed{a_n = 4/n^2} \rightarrow (2).$$

Step 4: To find b_n.

$u = (\pi - x)^2$ $\quad v = \sin nx$

$u' = 2(\pi - x)(-1)$ $\quad v_1 = -\frac{\cos n}{n}$

$u'' = 2$ $\quad v_2 = -\frac{\sin n}{n^2}$

$u''' = 0$ $\quad v_3 = \frac{\cos n}{n^3}$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2 \sin nx\,dx$$

$$= \frac{1}{\pi}\left[(\pi - x)^2\left(\frac{-\cos nx}{n}\right) - 2(\pi - x)(-1)\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos}{}\right.\right.$$

(using Bernoulli's form

$$= \frac{1}{\pi}\left[\frac{-\pi^2\cos 2n\pi}{n} + \frac{2\cos 2n\pi}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-\pi^2/n + 2/n^3 + \pi^2/n - 2/n^3\right] = 0$$

$(\because \cos 2n\pi$

$$\therefore \boxed{b_n = 0} \rightarrow (3)$$

Step 5: The required fourier series

from (1), (2) & (3) we get

$$a_0 = \frac{2\pi^2}{3}, \quad a_n = 4/n^2, \quad b_n = 0$$

substituting these values in (A) we

$$f(x) = \pi^2/3 + \sum_{n=1}^{\infty} 4/n^2 \cos nx.$$

$$f(x) = (\pi - x)^2 = \pi^2/3 + 4\left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3}{3^3}\right.$$

step 6: Deduction

Here 0 is a point of discontinuity which is an end point of the interval $0 < x < 2\pi$

The value of fourier series at $x=0$ is equal to the average value of $f(x)$ at the end points i.e. The series converges to $\frac{f(0)+f(2\pi)}{2}$

putting $x=0$ in (4), we get the fourier series,

$$\frac{\pi^2}{3}+4\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right] = \frac{f(0)+f(2\pi)}{2}$$

$$= \frac{\pi^2+\pi^2}{2} = \frac{2\pi^2}{2}$$

$$= \pi^2$$

i.e.,

$$\pi^2 = \frac{\pi^2}{3}+4\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \frac{3\pi^2-\pi^2}{12}$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \frac{\pi^2}{6}$$

Q:2 Expand $x(2\pi - x)$ as a fourier series in $(0, 2\pi)$ Deduce the sum of the series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$

soln:

step 1: Let $f(x) = x(2\pi - x)$

The fourier series of $f(x)$ in $(0, 2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$$

step 2: To find a_0.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x(2\pi - x)\,dx = \frac{1}{\pi}\int_0^{2\pi}(2\pi x - x^2)\,dx$$

$$= \frac{1}{\pi}\left[\pi x^2 - \frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{\pi}\left[4\pi^3 - \frac{8\pi^3}{3}\right]$$

$$= \frac{1}{\pi}\left[\frac{12\pi^3 - 8\pi^3}{3}\right] = \frac{1}{\pi}\cdot\frac{4\pi^3}{3} = \frac{4\pi^2}{3}$$

$$\boxed{a_0 = \frac{4\pi^2}{3}} \rightarrow (2)$$

step 3: To find a_n

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x(2\pi - x)\cos nx\,dx$$

Half Range Expansions

In many Engineering problems it is required to expand a function $f(x)$ in the range $(0, \pi)$ in a Fourier series of period 2π or in the range $(0, l)$ in a Fourier series of period $2l$. If it is required to expand $f(x)$ in the interval $(0, l)$, then it is immaterial what the function may be outside the range $0<x<l$. We are free to choose it arbitrarily in the interval $(-l, 0)$.

If we extend the function $f(x)$ by reflecting it in the Y axis so that $f(-x) = f$ then the extended function is even for which $b_n = 0$. The Fourier expansion of $f(x)$ will contain only cosine terms

If we extend the function $f(x)$ by reflecting it in the origin so that $f(-x) =$ then the extended function is odd for $a_0 = a_n = 0$ The Fourier expansion of $f(x)$ will contain only sine terms.

Hence a function $f(x)$ defined over the interval $0<x<l$ is capable of two distinct half range series

The half range cosine series in $(0, l)$ is
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$
where,
$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$
$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$
The half range sine series is
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$
where,
$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Note: (i) The half-range cosine series in $(0,\pi)$ is given by, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$,
where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

Note: (ii) The half range sine series in $(0,\pi)$ is given by, $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$,
where,
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

Problems on Fourier series sine in the interval $(0,\pi)$

Ex:1 Find the half-range sine series of $f(x)$ in $(0,\pi)$, given that $f(x) = \begin{cases} kx, & 0 \le x \le \pi/2 \\ k(\pi - x), & \pi/2 \le x \le \pi \end{cases}$

Soln:
Step 1: The half-range sine series of $f(x)$ in $(0,\pi)$ is given by
$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \to (1)$$

Step 2: To find b_n
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$
Now,
$$b_n = \frac{2}{\pi}\left[\int_0^{\pi/2} kx\sin nx\,dx + \int_{\pi/2}^{\pi} k(\pi - x)\sin nx\,dx\right.$$

$$= \frac{2}{\pi}\left[\left\{kx\left(\frac{-\cos nx}{n}\right) - k\left(\frac{-\sin nx}{n^2}\right)\right\}_0^{\pi/2} + \left\{k(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-k)\left(\frac{-\sin nx}{n^2}\right)\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{-k(\pi/2)\cos\frac{n\pi}{2}}{n} + \frac{k\sin\frac{n\pi}{2}}{n^2} + \frac{-k(\pi/2)\cos\frac{n\pi}{2}}{n} + \frac{k\sin n\pi/2}{n^2}\right]$$

$$= \frac{2}{\pi}\frac{2k\sin n\pi/2}{n^2}$$

$$\therefore \boxed{b_n = \frac{4k}{\pi n^2}\sin\frac{n\pi}{2}} \quad \rightarrow (2)$$

Step 3: The required sine series
substituting (2) in (1), we get

$$f(x) = \sum_{n=1}^{\infty}\frac{4k}{\pi n^2}\sin\frac{n\pi}{2}\sin nx$$

$$\therefore f(x) = \frac{4k}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}\sin nx}{n^2}$$

Ex: 2 Find a half-range sine series which represent $f(x) = \sin px$ for p not an integer on the interval $0 < x < \pi$

Solu:-

Step 1: The half-range sine series for $f(x)$ in $(0, \pi)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \rightarrow (1).$$

Step 2: To find b_n

Now,
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\sin px \sin nx\, dx$$

(Here note that 'p' is not an integer but 'n' is an integer)

$$= \frac{2}{\pi}\cdot\frac{1}{2}\int_0^{\pi}(\cos(n-p)x - \cos(n+p)x)\,dx$$

$(\because 2\sin A \sin B = \cos(A-B) - \cos(A+B))$

$$= \frac{1}{\pi}\left[\frac{\sin(n-p)x}{n-p} - \frac{\sin(n+p)x}{n+p}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(n-p)\pi}{n-p} - \frac{\sin(n+p)\pi}{n+p}\right] \quad (\because \sin 0 = 0)$$

$$= \frac{1}{\pi}\left[\frac{\sin n\pi \cos p\pi - \cos n\pi \sin p\pi}{n-p} - \frac{\sin n\pi \cos p\pi + \cos n\pi \sin p\pi}{n+p}\right]$$

$$= -\frac{1}{\pi}\left[\frac{\cos n\pi \sin p\pi}{n-p} + \frac{\cos n\pi \sin p\pi}{n+p}\right]$$

($\because \sin n\pi = 0$ since n is an integer p also $\sin p\pi \neq 0$ since p is not an integer)

$$= \frac{(-1)^{n+1}\sin p\pi}{\pi}\left[\frac{n+p+n-p}{n^2-p^2}\right]$$

$$= \frac{(-1)^{n+1}\sin p\pi}{\pi} \cdot \frac{2n}{n^2-p^2}$$

ie, $\boxed{b_n = \frac{2n}{\pi}\frac{(-1)^{n+1}\sin p\pi}{n^2-p^2}} \rightarrow (2)$

<u>step</u> 3: The required sine series substituting (2) in (1) $\Rightarrow$

$$f(x) = \sum_{n=1}^{\infty} \frac{2n(-1)^{n+1}\sin p\pi}{\pi(n^2-p^2)} \sin nx.$$

$$\therefore f(x) = \frac{2\sin p\pi}{\pi}\sum_{n=1}^{\infty}\frac{(n)(-1)^{n+1}\sin nx}{n^2-p^2}$$

<u>Ex: 3</u> Find the half range sine series of $f(x) = x\cos x$ in $(0, \pi)$

<u>Soln</u>:

<u>step</u> 1: The half range sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$$

<u>step</u> 2: To find b_n

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos x \sin nx\,d$$

$$= \frac{2}{\pi}\int_0^\pi x\left[\left(\frac{\sin(n+1)x + \sin(n-1)x}{2}\right)\right]dx$$

$$= \frac{1}{\pi}\int_0^\pi \left[x\sin(n+1)x + x\sin(n-1)x\right]dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos(n+1)x}{n+1}\right) - \left(\frac{-\sin(n+1)x}{(n+1)^2}\right) + x\left(\frac{-\cos(n-1)x}{n-1}\right) - \left(\frac{-\sin(n-1)x}{(n-1)^2}\right)\right]_0^\pi$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos(n+1)\pi}{n+1} - \frac{\pi\cos(n-1)\pi}{n-1}\right]$$

$$= -\left[\frac{\cos n\pi \cos\pi}{n+1} + \frac{\cos n\pi\cos\pi}{n-1}\right] \quad \left(\begin{array}{l}\because \sin(n+1)\pi = 0\\ \sin(n-1)\pi = 0\\ \sin n\pi = 0\end{array}\right)$$

$$= -\cos n\pi\cos\pi\left[\frac{1}{n+1} + \frac{1}{n-1}\right] \quad \left(\begin{array}{l}\because \cos n\pi = (-1)^n\\ \cos\pi = -1\end{array}\right)$$

$$b_n = (-1)^n\left[\frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= (-1)^n\left[\frac{n-1+n+1}{n^2-1}\right]$$

$$\therefore\ b_n = \frac{2n(-1)^n}{n^2-1},\ n \neq 1 \quad \rightarrow (2)$$

When $n = 1$,

$$b_1 = \frac{2}{\pi}\int_0^\pi x\cos x\sin x\,dx$$

$$= \frac{2}{\pi}\int_0^\pi \frac{x\sin 2x}{2}\,dx$$

$$= \frac{1}{\pi}\int_0^\pi x\sin 2x\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - \left(-\frac{\sin 2x}{4}\right)\right]_0^\pi$$

$$b_1 = -\frac{1}{2} \quad \rightarrow (3)$$

Step 3: The required sine series

substituting (2) & (3) in (1), we get

$$f(x) = b_1 \sin x + \sum_2^{\infty} b_n \sin nx$$

$$f(x) = \frac{-1}{2} \sin x + \sum_2^{\infty} \frac{2n(-1)^n}{n^2-1} \sin nx$$

Ex: Expand $f(x) = \cos x,\ 0 < x < \pi$ in a Fourier sine series

soln:

step 1: WKT, the fourier sine series of $f(x)$ in $0 < x < \pi$ is given by

$$f(x) = \sum_{n=1}^{\infty} a_n \sin nx \quad \rightarrow (1)$$

step 2: To find a_n

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi} \frac{1}{2}\left\{\sin(n+1)x + \sin(n-1)x\right\} dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[-\left(\frac{\cos n\pi \cos\pi - \sin n\pi \sin\pi}{n+1}\right) - \left(\frac{\cos n\pi \cos\pi + \sin n\pi \sin\pi}{n-1}\right) + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n+1} + \frac{\cos n\pi}{n-1} + \frac{1}{n+1} + \frac{1}{n+1}\right]$$

$(\because \sin n\pi = 0)$

$$= \frac{1}{\pi}\left[\frac{(n-1)\cos n\pi + (n+1)\cos n\pi + (n-1) + n+1}{(n+1)(n-1)}\right]$$

$$= \frac{1}{\pi}\left[\frac{2n\cos n\pi + 2n}{n^2-1}\right]$$

$$= \frac{2n}{\pi}\left[\frac{1+(-1)^n}{n^2-1}\right] \quad (\text{provided } n \neq 1)$$

$$\left.\begin{aligned} a_n &= 0, && \text{when 'n' is odd} \\ &= \frac{4n}{\pi(n^2-1)} && \text{when 'n' is even} \end{aligned}\right\} \quad -(2)$$

Now, $a_1 = \frac{2}{\pi}\int_0^{\pi} \cos x \sin x \, dx = \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx$

$$= \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = \frac{1}{\pi}\left[-\frac{1}{2}+\frac{1}{2}\right]$$

$$a_1 = 0 \rightarrow (3)$$

<u>Step 8</u>: The required Fourier sine series

substituting (2) & (3) in (1), we get

$$f(x) = a_1 \sin x + \sum_{n=2}^{\infty} a_n \sin nx$$

$$= 0 + \sum_{n=2,4}^{\infty} \frac{4n}{\pi(n^2-1)} \cdot \sin nx$$

$$= \frac{4}{\pi}\left[\frac{2}{3}\sin 2x + \frac{4}{15}\sin 4x + \frac{6}{35}\sin 6x + \right.$$

$$\cos x = \frac{8}{\pi}\left[\frac{\sin 2x}{3} + \frac{2}{15}\sin 4x + \frac{3}{35}\sin 6x + \cdots\right.$$

Problems on Fourier Cosine Series in the interval $(0,\pi)$

Ex: Find a cosine series for the function

$$f(x) = \begin{cases} x & \text{in } 0 \le x < \pi/2 \\ \pi - x & \text{in } \pi/2 \le x < \pi \end{cases}$$

Soln:

Step 1: The cosine series for the function $f(x)$ in $(0,\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \rightarrow (1)$$

Step 2: To find a_0

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi}(\pi - x)\,dx\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi/2} + \left(\pi x - \frac{x^2}{2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{8} + \left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right]$$

$$\boxed{a_0 = \pi/2} \rightarrow (2).$$

Step 3: To find a_n

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\left\{x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right\}_0^{\pi/2} + \left\{(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi/2 \cdot \sin n\pi/2}{n} + \frac{\cos n\pi/2}{n^2} - \frac{1}{n^2} - \frac{\cos n\pi}{n^2} - \frac{\pi/2\,\sin n\pi/2}{n} + \frac{\cos n\pi/2}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2\cos n\pi/2}{n^2} - \frac{1}{n^2} - \frac{\cos n\pi}{n^2}\right]$$

$$= \frac{2}{n^2\pi}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]$$

When n is odd,

$a_n = 0$, ie, $a_1 = a_3 = a_5 = 0 \rightarrow (3)$

When n is even

$$a_2 = \frac{2}{2^2\pi}\left[2\cos\pi - 1 - 1\right] = \frac{-2}{\pi\, 1^2} \rightarrow (4)$$

$$a_4 = \frac{2}{4^2\pi}\left[2\cos 2\pi - 1 - 1\right] = 0 \rightarrow (5)$$

$(\because \cos 2\pi = 1)$

$$a_6 = \frac{2}{6^2\pi}\left[2\cos 3\pi - 1 - 1\right] = \frac{-2}{\pi \cdot 3^2} \rightarrow (6)$$

& so on.

substituting (2), (3), (4), (5) & (6) in (1) we get

$$\therefore f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \cdots\right]$$

Ex: 2 Using an appropriate fourier expansion show that in the range $(0,\pi)$, the function $\sin x$ can be expressed as:

$$\frac{4}{\pi}\left(\frac{1}{2} - \frac{\cos 2x}{3} - \frac{\cos 4x}{15} - \frac{\cos 6x}{35} - \cdots - \frac{\cos 2nx}{4n^2-1}\right)$$

Soln:

Step 1: Here the range is $(0,\pi)$ and the expansion contains only cosine terms. Therefore we have to expand $\sin x$ in a half-range fourier cosine series in $(0,\pi)$.

WKT, the half-range cosine series of $f(x)$ in $(0,\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \rightarrow (1)$$

Step 2: To find a_0

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x\,dx$$

$$= \frac{2}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{2}{\pi}\left[-\cos\pi + \cos 0\right]$$

$$= \frac{2}{\pi}\left[1-(-1)\right] = 4/\pi \qquad (\because \cos\pi = -1)$$

$$\therefore \boxed{a_0 = 4/\pi} \rightarrow (2)$$

step 3: To find a_n

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\, dx$$

$$= \frac{2}{\pi}\cdot\frac{1}{2}\int_0^{\pi}(\sin(1+n)x + \sin(1-n)x)dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left[\frac{\cos(1+n)\pi}{1+n} + \frac{\cos(1-n)\pi}{1-n} - \frac{1}{1+n} - \frac{1}{1-n}\right]$$

When n is odd.

$$a_n = -\frac{1}{\pi}\left[\frac{1}{1+n} + \frac{1}{1-n} - \frac{1}{1+n} - \frac{1}{1-n}\right]$$

$\boxed{a_n = 0}$ ($\because$ when n is odd $\cos(1+n)\pi = 1 = \cos(1-n)\pi$)

provided $n \neq 1$ & n is odd $\to$ (3)

When $n = 1$,

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x\, dx = \frac{1}{\pi}\int_0^{\pi}\sin 2x\, dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi} = -\frac{1}{2\pi}[1-1] = 0$$

$\therefore \boxed{a_1 = 0} \to (4)$

When n is even, $(1+n)$ & $(1-n)$ is odd

$$a_n = -\frac{1}{\pi}\left[\frac{-1}{1+n} - \frac{1}{1-n} - \frac{1}{1+n} - \frac{1}{1-n}\right]$$

($\because \cos(1+n)\pi = 1; \cos(1-n)\pi = -1$, when 'n' is odd)

$$= \frac{1}{\pi}\left[\frac{2}{1+n} + \frac{2}{1-n}\right]$$

$$= \frac{2}{\pi}\left[\frac{1-n+1+n}{1-n^2}\right]$$

$$a_n = \frac{4}{\pi}\left[\frac{1}{1-n^2}\right] \to (5)$$

step 4: The required Cosine Series

substituting (2), (3), (4) & (5) in (1), we get

$$f(x) = \frac{2}{\pi} + \sum_{n=2,4,6}^{\infty} \frac{4}{\pi(1-n^2)}\cos nx.$$

$$= \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=2,4,6}^{\infty} \frac{1}{n^2-1} \cos nx$$

$$= \frac{4}{\pi} \left[\frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2-1} \cos 2nx \right]$$

i.e., $\sin x = \frac{4}{\pi}\left[\frac{1}{2} - \frac{\cos 2x}{3} - \frac{\cos 4x}{15} - \frac{\cos 6x}{35} - \cdots - \frac{\cos 2nx}{4n^2-1}\right.$

Ex: 3 show that in $0 \le x \le \pi$,

$$x(\pi - x) = \frac{\pi^2}{6} - \left(\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \cdots\right)$$

Soln:-

Step 1:

The half range cosine series of the function $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx. \quad -(1)$$

Here, $f(x) = x(\pi - x)$ (or) $\pi x - x^2$

Step 2: To find a_0

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\,dx$$

$$= \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right]$$

$$= \frac{2}{\pi} \cdot \frac{\pi^3}{6} = \frac{\pi^2}{3}$$

$$\therefore \boxed{a_0 = \pi^2/3} \rightarrow (2)$$

Step 3: To find a_n

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{\sin nx}{n}\right) - (\pi - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\pi \cos n\pi}{n^2} - \frac{\pi}{n^2}\right] = \frac{-2}{n^2}\left[1 + (-1)^n\right]$$

$$\therefore a_n = \begin{cases} 0, & \text{when } n \text{ is odd} \\ \frac{-4}{n^2}, & \text{when } n \text{ is even} \end{cases} \rightarrow (3)$$

Step 4: The required Fourier Cosine series

substituting (2) & (3) in (1), we get

$$f(x) = \frac{\pi^2}{6} + \sum_{n=2,4}^{\infty} \frac{-4}{n^2} \cdot \cos nx$$

$$= \frac{\pi^2}{6} - 4 \sum_{n=2,4}^{\infty} \frac{1}{n^2} \cdot \cos nx$$

$$x(\pi - x) = \frac{\pi^2}{6} - 4\left(\frac{\cos 2x}{2^2} + \frac{\cos 4x}{4^2} + \frac{\cos 6x}{6^2} + \cdots\right)$$