Differential Equation

Laplace Transforms (UNIT-IV)

Def:

Let $f(t)$ be a function of the variable t which is defined for all positive values of t. Let s be a real or complex parameter. If the integral (improper) $\int_0^\infty e^{-st} f(t)\,dt$ exists and is equal to $F(S)$ then $F(S)$ is called the Laplace Transform of $f(t)$ and it is denoted by the symbol $L\,f(t)$.

i.e., $$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \quad -(1)$$

$$= F(S)$$

$$\Rightarrow L\{f(t)\} = F(S)$$

As long as it is clear what is meant by the notation, this convention will facilitate our writing equations and carrying out calculation with Laplace transform.

Note 1: The integral $\int_0^\infty e^{-st} f(t)\,dt$ is a definite integral. If we integrate the above integral w.r.t. t and in the resulting solution if we are substituting upper and lower limits we get an expression which contains only s (s is parameter) and is denoted by F(s).

Note 2: The Laplace Transform of f(t) is said to exist if the integral given in (1) converges for some value of s.

Note 3: Here the operator L is called the Laplace transform operator which transforms the function f(t) into F(s).

Note 4: $\lim_{s \to \infty} F(s) = 0$

That is under the given condition the Laplace transform of t tends to zero as s approaches infinity.

Note 5: Not all function f(t) are Laplace transformable. For a function f(t) to be Laplace transformable it must satisfy the following sufficient conditions which is given below.

Sufficient condition for existence of Laplace transforms.

piecewise continuous function:

A function f(t) is said to be piecewise continuous in any closed interval interval [a, b], if it is defined on that interval and is such that the interval can be broken up into a finite number of sub intervals in each of which f(t) is continuous.

Function of exponential order

A function f(t) is said to be of exponential order if

$$\lim_{t \to \infty} e^{-st} f(t) = 0$$

Eg: The function t^2 is of exponential order

For, $\lim_{t \to \infty} e^{-st} t^2 = \lim_{t \to \infty} \frac{t^2}{e^{st}} \left(\frac{\infty}{\infty}\right) (s>0)$

Recall L'H rule

$$= \lim_{t \to \infty} \frac{2t}{s e^{st}} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{t \to \infty} \frac{2}{s^2 e^{st}} = 0 \text{ (fi}$$

$\therefore \lim_{t \to \infty} e^{-st} t^2 = 0$ (finite number)

$\therefore t^2$ is of exponential order

Note: The function such as e^{t^2} etc. are not Laplace transformable.

The sufficient condition for the existence of the Laplace transform are

(i) $f(t)$ should be continuous or piecewise continuous in the given closed interval $[a, b]$ where $a > 0$.

(ii) $f(t)$ should be of exponential order.

Important Results

Result 1: P.T. $L[e^{at}] = \frac{1}{s-a}$ provided $s-a>0$

Soln:

$$L[e^{at}] = \int_0^\infty e^{-st} \cdot e^{at}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty \quad \left(\because \int e^{ax} dx = \frac{e^{ax}}{a}\right)$$

$$= \frac{e^{-\infty}}{-(s-a)} + \frac{e^{0}}{s-a}$$

$$= \frac{1}{s-a}$$

$$\therefore L[e^{at}] = \frac{1}{s-a}$$

Ex 1: $L[e^{2t}] = ?$

Here $a = 2$

WKT $L[e^{at}] = \frac{1}{s-a}$

$\therefore L[e^{2t}] = \frac{1}{s-2}$

Ex 2: $L[e^{3/2\, t}] = ?$

Here $a = 3/2$

$L[e^{3/2\, t}] = \frac{1}{s-3/2}$

Ex 3: $L[e^{\sqrt{3}t}] = ?$

Here $a = \sqrt{3}$

$L[e^{\sqrt{3}t}] = \frac{1}{s-\sqrt{3}}$

Ex 4: $L[e^{1.7t}] = ?$

Here $a = 1.7$

$L[e^{1.7t}] = \frac{1}{s-1.7}$

Ex 5: $L[e^{0.001t}] = ?$

Here, $a = 0.001$

$\therefore L[e^{0.001t}] = \frac{1}{s-0.001}$

Ex 6: $L[e^{100002t}] = ?$

Here $a = 100002$

$\therefore L[e^{100002t}] = \frac{1}{s-100002}$

Ex 7: $L\left[e^{1/\sqrt{n}\, t}\right] = ?$

Here, $a = 1/\sqrt{6}$

$$L\left[e^{-1/\sqrt{6}\,t}\right] = \frac{1}{s+1/\sqrt{6}}$$

Result 2

P.T $\quad L\left[e^{-at}\right] = \frac{1}{s+a}$ provided

Solu:

$$L\left[e^{-at}\right] = \int_0^\infty e^{-st} \cdot e^{-at}\, dt = \int_0^\infty e^{-(s+a)t}\,dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{e^{-\infty}}{-(s+a)} - \frac{e^{0}}{-(s+a)}$$

$$= \frac{1}{s+a} \qquad \left[\because e^{-\infty} = 0,\ e^{0} = 1\right]$$

$$\therefore L\left[e^{-at}\right] = \frac{1}{s+a}$$

Eg 1 $\quad L\left[e^{-7t}\right] = ?$

Here $a = 7$

WKT, $L\left[e^{-at}\right] = \frac{1}{s+a}$

$$\therefore L\left[e^{-7t}\right] = \frac{1}{s+7}$$

Eg 2 $\quad L\left[e^{-5/3\,t}\right] = ?$

Here $a = 5/3$

$$\therefore L\left[e^{-5/3\,t}\right] = \frac{1}{s+5/3}$$

Eg 3 $\quad L\left[e^{-\sqrt{7}\,t}\right] = ?$

Here, $a = \sqrt{7}$

$$\therefore L\left[e^{-\sqrt{7}\,t}\right] = \frac{1}{s+\sqrt{7}}$$

Eg 4 $\quad L\left[e^{-10.05\,t}\right] = ?$

Here, $a = 10.05$

$$\therefore L\left[e^{-10.05\,t}\right] = \frac{1}{s+10.05}$$

Eg 5 $\quad L\left[e^{-9840174086\,t}\right] = ?$

Here, $a = 9840174086$

$$\therefore L\left[e^{-9840174086\,t}\right] = \frac{1}{s+9840174086}$$

Eg 6 $\quad L\left[e^{-\sqrt{7/11}\,t}\right] = ?$

Here $a = \sqrt{7/11}$

$$\therefore L\left[e^{-\sqrt{7/11}\,t}\right] = \frac{1}{s+\sqrt{7/11}}$$

Eg 7 $\quad L\left[e^{-0.005\,t}\right] = ?$

Here $a = 0.005$

$$\therefore L\left[e^{-0.005\,t}\right] = \frac{1}{s+0.005}$$

Result 3

P.T $\quad L\left[\cosh at\right] = \frac{s}{s^2-a^2}$

Solu: $\quad L\left[\cosh at\right] = L\left[\frac{e^{at}+e^{-at}}{2}\right]$

$= \frac{1}{2}\{L(e^{at}) + L(e^{-at})\}$

$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$

$= \frac{1}{2}\left[\frac{s+a+s-a}{(s-a)(s+a)}\right]$

$= \frac{1}{2} \cdot \frac{2s}{s^2-a^2} = \frac{s}{s^2-a^2}$

$\therefore L[\cosh at] = \frac{s}{s^2-a^2}$

problems

Eg: 1 $L[\cosh 7t] = ?$ WKT, $L[\cosh at] = \frac{s}{s^2-a^2}$

Here, $a = 7$

$L[\cosh 7t] = \frac{s}{s^2-(7)^2}$

$= \frac{s}{s^2-49}$

Eg: 2 $L[\cosh \sqrt{3}t] = ?$

Here $a = \sqrt{3}$

$\therefore L[\cosh \sqrt{3}t] = \frac{s}{s^2-(\sqrt{3})^2} = \frac{s}{s^2-3}$

Eg: 3 $L[\cosh (1+\sqrt{5})t] = ?$

Here, $a = 1+\sqrt{5}$

$L[\cosh (1+\sqrt{5})t] = \frac{s}{s^2-(1+\sqrt{5})^2}$

Eg 4 $L[\cosh 10.1t] = ?$

Here $a = 10.1$

$\therefore L[\cosh 10.1t] = \frac{s}{s^2-(10.1)^2}$

Eg: 5 $L[\cosh 3/2\, t] = ?$

Here, $a = 3/2$

$\therefore L[\cosh 3/2\, t] = \frac{s}{s^2-(3/2)^2} = \frac{s}{s^2-9/4}$

Result 4: P.T $L[\sinh at] = \frac{a}{s^2-a^2}$

soln:

$L[\sinh at] = L\left[\frac{e^{at}-e^{-at}}{2}\right]$

$= \frac{1}{2}\{L(e^{at}) - L(e^{-at})\}$

$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$

$= \frac{1}{2}\left[\frac{s+a-s+a}{(s-a)(s+a)}\right]$

$= \frac{1}{2} \cdot \frac{2a}{s^2-a^2} = \frac{a}{s^2-a^2}$

$\therefore L[\sinh at] = \frac{a}{s^2-a^2}$

problems

Eg: 1 $L[\sinh 17t] = ?$

Here, $a = 17$

WKT, $L[\sinh at] = a/s^2-a^2$

$\therefore L[\sinh 17t] = \frac{17}{s^2-(17)^2}$

$= \frac{17}{s^2-289}$

Eg: 2 $L[\sinh\sqrt{5}t] = ?$

Here, $a = \sqrt{5}$

$\therefore L[\sinh\sqrt{5}t] = \frac{\sqrt{5}}{s^2-(\sqrt{5})^2}$

$= \frac{\sqrt{5}}{s^2-5}$

Eg: 3 $L[\sinh 7.5t] = ?$

Here, $a = 7.5$

$L[\sinh 7.5t] = \frac{7.5}{s^2-(7.5)^2}$

Eg: 4 $L[\sinh {}^{15}/_{17}t] = ?$

Here, $a = 15/17$

$\therefore L[\sinh {}^{15}/_{17}] = \frac{\frac{15}{17}}{s^2-({}^{15}/_{17})^2}$

Eg: 5 $L[\sinh 10000t] = ?$

Here, $a = 10000$

$\therefore L[\sinh 10000t] = \frac{10000}{s^2-(10000)^2}$

Result 5: P.T $L[\cos at] = \frac{s}{s^2+a^2}$

solution:

Method - 1

$L[\cos at] = \int_0^\infty e^{-st}\cos at\, dt$

$= \left[\frac{e^{-st}(-s\cos at + a\sin at)}{s^2+a^2}\right]_0^\infty$

$\left[\because \int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$

$a = -s,\ b = a,\ t = x$

$= \frac{e^{-\infty}}{s^2+a^2} - \frac{e^0(-s)}{s^2+a^2}$

$= 0 + \frac{s}{s^2+a^2}$ $(\because e^{-\infty}=0, e^0=1)$

$= \frac{s}{s^2+a^2}$

$\therefore L[\cos at] = s/s^2+a^2$

Method - 2

P.T $\Rightarrow L[\cos at] = s/s^2+a^2$

$L[\cos at] = \int_0^\infty e^{-st}\cos at\, dt$

By the def of $L(t)$.

$= R.P \text{ of } \int_0^\infty e^{-st}(e^{iat})\, dt$

$= R.P \text{ of } L[e^{iat}]$

$= R.P \text{ of } \frac{1}{s-ia} = R.P \text{ of } \frac{s+ia}{(s-ia)(s+ia)}$

$= R.P \text{ of } \frac{s+ia}{s^2+a^2}$

$= R.P \text{ of } \frac{s}{s^2+a^2}$

$\therefore L[\cos at] = \frac{s}{s^2+a^2}$

Problems for Result 6

Eg 1 $L[\sin 7t] = ?$ W.k.t $L[\sin at] = a/s^2+a^2$

Here, $a = 7$

$L[\sin 7t] = 7/s^2+(7)^2$

$= 7/s^2+49$

Eg 2 $L[\sin 1.3\,t] = ?$

Here, $a = 1.3$

$L[\sin 1.3\,t] = \frac{1.3}{s^2+(1.3)^2}$

Eg 3 $L[\sin \sqrt{10}\,t] = ?$

Here, $a = \sqrt{10}$

$L[\sin \sqrt{10}\,t] = \frac{\sqrt{10}}{s^2+(\sqrt{10})^2}$

$= \frac{\sqrt{10}}{s^2+10}$

Eg 4 $L[\sin 1000t] = ?$

Here, $a = 1000$

$L[\sin 1000t] = \frac{1000}{s^2+(1000)^2}$

Result - 6

P.T $L[\sin at] = a/s^2+a^2$

Soln:

Method - 1

$L[\sin at] = \int_0^\infty e^{-st} \sin at\, dt$

$= \left[\frac{e^{-st}(-s\sin at - a\cos at)}{s^2+a^2}\right]_0^\infty$

$\left[\because \int_0^\infty e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$

$= \frac{e^{-\infty}}{s^2+a^2} - \frac{e^0(-a)}{s^2+a^2}$

$= \frac{a}{s^2+a^2}$ $[\because e^{-\infty}=0,\ e^0=1]$

$\therefore L[\sin at] = \frac{a}{s^2+a^2}$

Method 2:

$L[\sin at] = \int_0^\infty e^{-st}\sin at\, dt$

$=$ Im part of $\int_0^\infty e^{-st}.e^{iat}\, dt$

$=$ I.P of $L[e^{iat}]$

$=$ I.P of $1/s-ia$

$=$ I.P of $\frac{s+ia}{(s-ia)(s+ia)}$

$=$ I.P of $\frac{s+ia}{s^2+a^2}$

$=$ I.P of $\frac{a}{s^2+a^2}$

$\therefore L[\sin at] = \frac{a}{s^2+a^2}$

problem for Result 5

Eg: 1 $L[\cos 100t] = ?$

Here, $a = 100$

$L[\cos 100t] = \frac{s}{s^2+(100)^2}$

$= \frac{s}{s^2+10000}$

Eg: 2 $L[\cos \sqrt{17}\, t] = ?$

Here, $a = \sqrt{17}$

$\therefore L[\cos \sqrt{17}\, t] = \frac{s}{s^2+(\sqrt{17})^2}$

$= \frac{s}{s^2+17}$

Eg: 3 $L[\cos 1.732\, t] = ?$

Here, $a = 1.732$

$\therefore L[\cos 1.732\, t] = \frac{s}{s^2+(1.732)^2}$.

Eg: 4 $L[\cos 7/2\, t] = ?$

Here, $a = 7/2$

$\therefore L[\cos 7/2\, t] = \frac{s}{s^2+(7/2)^2}$

Eg: 5 $L[\cos \sqrt{5/2}\, t] = ?$

Here, $a = \sqrt{5/2}$

$\therefore L[\cos \sqrt{5/2}\, t] = \frac{s}{s^2+(\sqrt{5/2})^2} = \frac{s}{s^2+5/2}$

Result - 7

P.T $L[K] = K/s$

proof: $L[K] = L[Ke^{0t}]$

$= \int_0^\infty Ke^{-st} \cdot e^{0t}\, dt$

$= K\left[\frac{e^{-st}}{-s}\right]_0^\infty$

$= K\,\frac{e^{-\infty} - e^0}{-s} = K/s$

$\therefore L[K] = K/s$

when $k = 1$ we have $L(1) = 1/s$

problem

Eg: 1 $L[2] = 2/s$

$L[1.7] = 1.7/s$

$L[100] = 100/s$

$L[\sqrt{3}/2] = \sqrt{3}/2 \cdot 1/s$

$L[2010] = \frac{2010}{s}$

Linear property

If c_1 & c_2 are constant and $f_1(t)$ & $f_2(t)$ are given functions, then $L[c_1 f_1(t) + c_2 f_2(t)] = c_1 L[f_1(t)] + c_2 L[f_2(t)]$.

proof:

$L[c_1 f_1(t) + c_2 f_2(t)] = \int_0^\infty e^{-st} \{c_1 f_1(t) + c_2 f_2(t)\} dt$

$= \int_0^\infty \{c_1 e^{-st} f_1(t) + c_2 e^{-st} f_2(t)\} dt$

$= c_1 \int_0^\infty e^{-st} f_1(t)\, dt + c_2 \int_0^\infty e^{-st} f_2(t)\, dt$

$= c_1 L[f_1(t)] + c_2 L[f_2(t)]$

Hence the linear property

Solved problems using linear property

Ex: Find $L(e^{2t} + 3e^{-4t})$

solu:

$L(e^{2t} + 3e^{-4t})$ $\quad (\because L(e^{at}) = \frac{1}{s-a}$, $L(e^{-at}) = \frac{1}{s+a})$

$= L(e^{2t}) + 3L(e^{-4t})$

$= \frac{1}{s-2} + 3 \cdot \frac{1}{s+4}$

Ex: 2 Find $L(3e^{5t} + 5\cos t)$

solu:

$L(3e^{5t} + 5\cos t)$ $\quad (\because L(e^{at}) = \frac{1}{s-a}$, $L(\cos at) = \frac{s}{s^2+a^2})$

$= 3L(e^{5t}) + 5L(\cos t)$

$= 3 \cdot \frac{1}{s-5} + 5 \cdot \frac{s}{s^2+1}$

Ex: 3 Find $L(\sinh 6t + 3e^{-5t} + \cos 5t)$

solu:

$L(\sinh 6t + 3e^{-5t} + \cos 5t)$

$= L(\sinh 6t) + 3L(e^{-5t}) + L(\cos 5t)$ $\quad (\because L[\sinh at] = \frac{a}{s^2-a^2}$, $L[e^{-at}] = \frac{1}{s+a}$, $L[\cos at] = \frac{s}{s^2+a^2})$

$= \frac{6}{s^2-6^2} + 3 \cdot \frac{1}{s+5} + \frac{s}{s^2+5^2}$

$= \frac{6}{s^2-36} + \frac{3}{s+5} + \frac{s}{s^2+25}$

Ex: 4

Find $L(\sin^2 2t)$ $\quad [\because \sin^2 x = \frac{1-\cos 2x}{2}]$

solu:

$L(\sin^2 2t) = L\left[\frac{1-\cos 4t}{2}\right]$

$= L(1/2) - L\left[\frac{\cos 4t}{2}\right]$

$= 1/2\, L(1) - 1/2\, L[\cos 4t]$ $\quad [\because L(k) = k/s$, $L(\cos at) = \frac{s}{s^2+a^2}]$

$= 1/2s - 1/2 \cdot \frac{s}{s^2+4^2}$

$= 1/2s - 1/2 \cdot \frac{s}{s^2+16}$

Ex: 5

Find $L(\cos^2 3t)$ $\quad [\because \cos^2 x = \frac{1+\cos 2x}{2}]$

solu:

$L(\cos^2 3t) = L\left[\frac{1+\cos 6t}{2}\right]$

$= 1/2 \{L(1) + L(\cos 6t)\}$

$$= \frac{3}{4}\left(\frac{3}{s^2+9}\right) - \frac{1}{4}\left(\frac{9}{s^2+81}\right) + \frac{3}{4}\left(\frac{s}{s^2-9}\right)$$

$$+ \frac{1}{4}\left(\frac{s}{s^2-81}\right)$$

Result : 8 P.T $L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$

(or) $L[t^n] = \frac{n!}{s^{n+1}}$

put $st = x$ when $t=0,\ x=0$

$\therefore s\,dt = dx$ when $t=\infty,\ x=\infty$

$$\therefore L(t^n) = \int_0^\infty e^{-x}(x/s)^n \, 1/s \, dx$$

$$= \frac{1}{s^{n+1}} \int_0^\infty x^n e^{-x}\, dx$$

$$L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$$

$\left(\because \int_0^\infty x^n e^{-x} dx = \Gamma(n+1)\right)$

When n is a positive integer,

then $\Gamma(n+1) = n!$

$\therefore L(t^n) = \frac{n!}{s^{n+1}}$ (n is a positive integer).

Ex (i) $L(1) = L(t^0)$

$= \frac{0!}{s^{0+1}} = 1/s$

$\therefore L(1) = 1/s$

Ex (ii) $L(t) = \frac{1!}{s^{1+1}} = 1/s^2$

$\therefore L(t) = 1/s^2$

Ex (iii) $L(t^2) = \frac{2!}{s^{2+1}} = 2/s^3$

$\therefore L(t^2) = 2/s^3$

Ex (iv) $L(t^3) = \frac{3!}{s^{3+1}}$

$\therefore L(t^3) = 6/s^4$

Ex (v) $L[\sqrt{t}] = L(t^{1/2})$

$= \frac{\Gamma(1/2+1)}{s^{1/2+1}}$ (Here n is not an integer)

$= \frac{1/2\,\Gamma 1/2}{s^{3/2}}$

$\therefore L(t^{1/2}) = \sqrt{\pi} / 2s^{3/2}$

Ex (vi) $L(t^{-1/2}) = \frac{\Gamma(-1/2+1)}{s^{-1/2+1}}$

$= \frac{\Gamma 1/2}{s^{1/2}} = \frac{\sqrt{\pi}}{s^{1/2}}$

$\therefore L(t^{-1/2}) = \sqrt{\pi/s}$

Ex (vii) $L[t^{10}] = \frac{10!}{s^{10+1}}$

$\therefore L[t^{10}] = \frac{10!}{s^{11}}$

Ex (viii) $L[t^{125}] = \frac{125!}{s^{125+1}}$

$\therefore L[t^{125}] = \frac{125!}{s^{126}}$

Ex (ix) $L[t^{3/2}] = \frac{\Gamma(3/2+1)}{s^{3/2+1}}$

$= \frac{3/2\,\Gamma 3/2}{s^{5/2}}$

$= 3/2\,\frac{\Gamma(1/2+1)}{s^{5/2}}$

$= 3/2 \cdot 1/2\,\frac{\Gamma(1/2)}{s^{5/2}}$

$L[t^{3/2}] = \frac{3}{4} \cdot \frac{\sqrt{\pi}}{s^{5/2}}$

$(\because \Gamma(1/2) = \sqrt{\pi})$

Solved problems using properties (1) to (8)

Ex: Find $L(a + bt + c/\sqrt{t})$

solu: $L(a+b+c/\sqrt{t}) = L(a) + L(b) + L\left(\frac{c}{\sqrt{t}}\right)$

$= a\,L(1) + b\,L(t) + c\,L(t^{-1/2})$

$= a/s + b/s^2 + c\sqrt{\pi/s}$

$(\because L(K) = K/s$

$L(t) = 1/s^2$

$L(t^{-1/2}) = \sqrt{\pi/s}$

Ex 2 Find $L(5 - 3t - 2e^{-t})$

Solu:

$L(5 - 3t - 2e^{-t}) = L(5) - L(3t) - L(2e^{-t})$

$(\because L(K) = K/s$ $\quad = L(5) - 3 \cdot L(t) - 2L(e^{-t})$

$L(t) = 1/s^2$

$L(e^{-at}) = 1/s+a)$ $\quad = 5/s - 3/s^2 - 2/s+1$

$$= \frac{3s^2 + 2s - 3}{s^2(s+1)}$$

Ex:3 Find $L[(\alpha + \beta t)^2]$

Solu:

$L[(\alpha + \beta t)^2] = L[\alpha^2 + \beta^2 t^2 + 2\alpha\beta t]$

$(\because L(K) = K/s$ $\quad = L[\alpha^2] + L[\beta^2 t^2] + L[2\alpha\beta t]$

$L(t^2) = 2!/s^3$

$L(t) = 1/s^2)$ $\quad = L[\alpha^2] + \beta^2 L[t^2) + 2\alpha\beta\, L[t]$

$$= \frac{\alpha^2}{s} + \beta^2 \frac{2!}{s^3} + 2\alpha\beta \frac{1}{s}$$

Ex:4 Find $L[e^{-8t} + \cosh 2t + \sin 7t]$.

Solu:-

$L[e^{-8t} + \cosh 2t + \sin 7t]$

$= L[e^{-8t}] + L[\cosh 2t] + L[\sin 7t]$

$$= \frac{1}{s+8} + \frac{s}{s^2-4} + \frac{7}{s^2+49}$$

$(\because L(e^{-at}) = \frac{1}{s+a}$

$L(\cosh at) = \frac{s}{s^2-a^2}$

$L(\sin at) = \frac{a}{s^2+a^2})$

Ex 5

Find $L(7e^{2t} + 9e^{-2t} + 5\cos t + 7t^3 + 5\sin 3t + 2)$.

Solu:

$L(7e^{2t} + 9e^{-2t} + 5\cos t + 7t^3] + 5\sin 3t + 2$

$= L(7e^{2t}) + L(9e^{-2t}) + L(5\cos t) + L(7t^3) + L(5\sin 3t) + L(2)$

$(\because L(e^{at}) = 1/s-a, \quad L(e^{-at}) = 1/s+a,$

$L(t^3) = 6/s^4, \quad L[\sin at] = a/s^2+a^2,$

$L(\cos at) = s/s^2+a^2, \quad L(K) = K/s)$

$= 7L(e^{2t}) + 9L(e^{-2t}) + 5L(\cos t) + 7L(t^3) + 5L(\sin 3t) + L(2)$

$$= \frac{7}{s-2} + \frac{9}{s+2} + 5 \cdot \frac{s}{s^2+1} + 7 \cdot \frac{3!}{s^4} + 5 \cdot \frac{3}{s^2+9} + \frac{2}{s}$$

Ex:6

Find $L[(t+1)^2]$.

Solu:

$L[(t+1)^2] = L[t^2 + 2t + 1]$

$= L(t^2) + L(2t) + L(1)$

$(\because L(t^2) = \frac{2!}{s^3} \quad L(t) = 1/s^2 \quad L(K) = K/s)$

First shifting theorem (First translation)
(shifting in the s-variable)

If $L[f(t)] = F(s)$, then $L[e^{at} f(t)] = F(s-a)$

That is the Laplace transform of $e^{at} f(t)$ is obtained by replacing s by $s-a$ in the Laplace transform of f. We also can say, the effect of multiplying f by e^{at} is to shift the Laplace transform of f by a units to the right.

proof: $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s)$

$$\therefore L[e^{at} f(t)] = \int_0^\infty e^{-st} [e^{at} f(t)]\, dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt, \quad s-a>0$$

$$= F(s-a)$$

$$\therefore L[e^{at} f(t)] = F(s-a)$$

where $F(s) = L[f(t)]$

Corollary: $L[e^{-at} f(t)] = F(s+a)$,

where $F(s) = L[f(t)]$

Solved problems using First shifting theorem.

Ex: 1 Find $L\left[\sum_{n=0}^{N} a_n e^{-bt} \cos nt\right]$

Solu:-

$L\left[\sum_{n=0}^{N} a_n e^{-bt} \cos nt\right]$

$= L[a_0 e^{-bt}] + L[a_1 e^{-bt} \cos t] +$

$L[a_2 e^{-bt} \cos 2t]$

$(\because L[e^{-at}] = 1/s+a \quad , \quad L[\cos at] = \frac{s}{s^2+a^2}$

$= a_0 \frac{1}{s+b} + a_1 \frac{s+b}{(s+b)^2+1^2} + a_2 \frac{s+b}{(s+b)^2+2^2}$

$+ \dots$

$= \sum_{n=0}^{N} a_n \frac{s+b}{(s+b)^2+n^2}$

Ex: 2 Find $L[e^{-3t} \sin^2 t]$

solu:-

Wkt,

$L[e^{-at} f(t)] = F(s+a)$,

where $F(s) = L[f(t)]$

$\therefore L[f(t)] = L[\sin^2 t]$

$= L\left[\frac{1-\cos 2t}{2}\right]$

$= 1/2 \,[L(1) - L(\cos 2t)]$

$(\because L(K) = K/s \quad , \quad L(\cos at) = s/s^2+a^2)$

$= 1/2 \,[1/s - s/s^2+4] \; -(1)$

$= 1/2 \left[\frac{1}{s+3} - \frac{s+3}{(s+3)^2+4}\right]$

[Replacing s by s+3 in (1)]

$\therefore L[e^{-3t} \sin^2 t] = 1/2 \left[\frac{1}{s+3} - \frac{s+3}{(s+3)^2+4}\right]$

Ex: 3 Find $L[e^{t} (\cosh 2t + 1/2 \sinh 2t)]$

solu:-

Wkt, $L[e^{at} f(t)] = F(s-a)$

Here, $f(t) = [\cosh 2t + 1/2 \sinh 2t]$;

$F(s) = L[f(t)]$

$\therefore L[f(t)] = L[\cosh 2t + 1/2 \sinh 2t]$

$= L[\cosh 2t] + 1/2\, L[\sinh 2t]$

$(\because L[\cosh at] = s/s^2-a^2 \quad , \quad L[\sinh at] = a/s^2-a^2)$

$= \frac{s}{s^2-4} + 1/2 \cdot \frac{2}{s^2-4} \; -(1)$

$\therefore L[e^{t} (\cosh 2t + 1/2 \sinh 2t)]$

$= \frac{s-1}{(s-1)^2-4} + \frac{1}{(s-1)^2-4}$

(Replacing s by s-1 in (1))

$$\left[\frac{\quad}{s^{1/2}}\right]_{s\to s+7}$$

$$= \left[\frac{\sqrt{\pi}}{\sqrt{s}}\right]_{s\to s+7} = \frac{\sqrt{\pi}}{\sqrt{s+7}}$$

Ex: 11 Find $L[e^{100t} \cdot t^{10}]$

solu:

$$L[e^{100t} \cdot t^{10}] = L[t^{10}]_{s\to s-100}$$

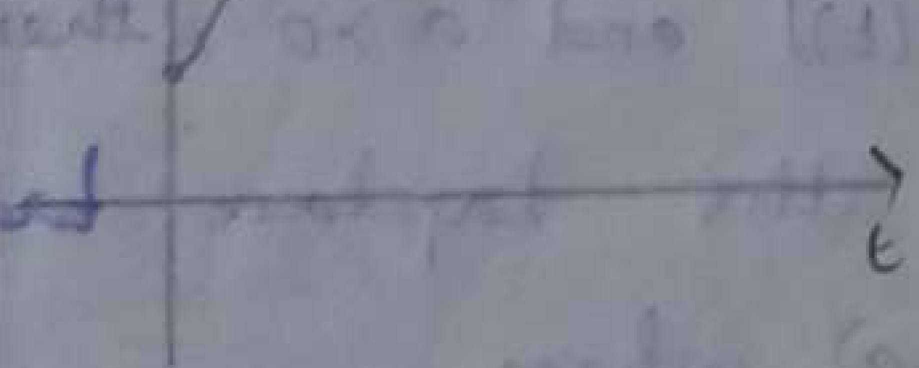
$$= \left(\frac{10!}{s^{10+1}}\right)_{s\to s-100}$$

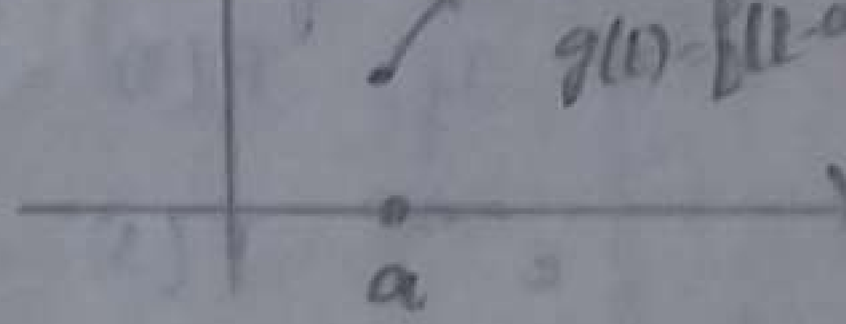
$$= \frac{10!}{(s-100)^{11}}$$

Second shifting theorem (second Translation)

If $L[f(t)] = F(s)$ and $G(t) = \begin{cases} f(t-a), & t>a \\ 0, & 0\le t<a \end{cases}$

then $L[G(t)] = e^{-as} F(s)$.





Proof:

$$L[G(t)] = \int_0^\infty e^{-st} G(t)\,dt.$$

$$= \int_0^\infty e^{-st} G(t)\,dt + \int_a^\infty e^{-st} G(t)\,dt$$

$$= \int_0^a e^{-st} \cdot 0 \cdot dt + \int_a^\infty e^{-st} f(t-a)\, dt$$

$$= 0 + \int_a^\infty e^{-st} f(t-a)\, dt$$

put $t - a = u$ when $t = a$, $u = 0$

$dt = du$ $t = \infty$, $u = \infty$

$$\therefore L(G(t)) = \int_0^\infty e^{-s(u+a)} \cdot f(u)\, du$$

$$= e^{-sa} \int_0^\infty e^{-su} f(u)\, du$$

In $\int_0^\infty e^{-su} f(u)\, du$, u is dummy variable

Hence we can replace it by the variable t.

$$\therefore L[G(t)] = e^{-sa} \int_0^\infty e^{-st} f(t)\, dt$$

$$= e^{-sa} L[f(t)]$$

$$= e^{-sa} F(s)$$

Another form of the second shifting Theorem:

If $F(s) = L[F(t)]$ and $a > 0$, then $e^{-as} \cdot F(s)$ is the Laplace transform of $F(t-a) \cdot H(t-a)$, where

$$H(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$$

i.e., $L[F(t-a)\, H(t-a)] = e^{-as} \cdot F(s)$

Proof:

8.

9. t^n

Solved problems using second shifting theorem.

Ex: 1 Find the Laplace Transform of $G(t)$, where

$$G(t) = \begin{cases} \cos(t - 2\pi/3) & \text{if } t > 2\pi/3 \\ 0 & \text{if } t < 2\pi/3 \end{cases}$$

Soln:

WKT by second shifting property

if $\quad L[f(t)] = F(s) \quad$ &

$$G(t) = \begin{cases} F(t-a) & , t > a \\ 0 & , t < a \end{cases}$$

then $\quad L[G(t)] = e^{-at} F(s) \quad -(1)$

Here

$$F(t-a) = \cos(t - 2\pi/3)$$

ie, $F(t) = \cos t \quad$ & $\quad a = 2\pi/3$.

$$\therefore L[F(t)] = L[\cos t] = s/s^2+1 \quad -(3)$$

substituting (2) & (3) in (1) we get,

$$\therefore L[G(t)] = e^{-\frac{2\pi s}{3}} \cdot \frac{s}{s^2+1}$$

4 Find $L[f(t)]$, where $f(t) = \begin{cases} 0, & \text{when } 0<t<2 \\ 3, & \text{when } t>2 \end{cases}$

WKT,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^2 e^{-st} f(t)\,dt + \int_2^\infty e^{-st} f(t)\,dt$$

$$= \int_0^2 e^{-st} \cdot 0 \cdot dt + \int_2^\infty e^{-st} \cdot 3\,dt$$

$[\because f(t) = 0$ in $(0,2)$

$f(t) = 3$ in $(2,\infty)]$

$$= 3\left[\frac{e^{-st}}{-s}\right]_2^\infty = 3\left[\frac{e^{-\infty} - e^{-2s}}{-s}\right]$$

$$= \frac{3e^{-2s}}{s}$$

5 Find the Laplace transform of

$$f(t) = \begin{cases} \sin t, & 0<t<\pi \\ 0, & t>\pi \end{cases}$$

Soln: WKT,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\pi e^{-st} f(t)\,dt + \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\pi e^{-st} \sin t\,dt + \int_\pi^\infty e^{-\pi} \cdot 0 \cdot dt$$

$(\because f(t) = \sin t$ in $(0,\pi)$

$f(t) = 0$ in $(\pi,\infty)]$

$$= \left[\frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\right]_0^\pi$$

$$= \frac{e^{-s\pi}}{s^2+1}\left(-s\sin\pi t - \cos\pi t\right) + \frac{\ldots}{s^2+1}$$

$$= \frac{e^{-\pi s}}{s^2+1} + \frac{1}{s^2+1} + \frac{1+e^{-\pi s}}{s^2+1}$$

$(\because \sin n\pi = 0)$

Ex: 6

Find the Laplace transform of $f(t)$, where $f(t) = \begin{cases} e^t, & 0<t<1 \\ 0, & t>1 \end{cases}$

Soln:

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st} f(t)\,dt + \int_1^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st} \cdot e^t\,dt + \int_1^\infty e^{-st} \cdot 0 \cdot dt$$

$$= \int_0^1 e^{-(s-1)t}\,dt$$

$$= \left[\frac{e^{-(s-1)t}}{-(s-1)}\right]_0^1$$

$$= \frac{e^{-(s-1)}}{-(s-1)} + \frac{e^0}{s-1}$$

$$= \frac{1-e^{-(s-1)}}{s-1}$$

Ex: 7 Find $L[f(t)]$, where

$$f(t) = \begin{cases} \cos t, & \text{when } 0<t<\pi \\ \sin t, & \text{when } t>\pi \end{cases}$$

Soln: $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^\pi e^{-st}\cos t\,dt + \int_\pi^\infty e^{-st}\sin t\,dt$$

$$= \left[\frac{e^{-st}}{s^2+1}\left(-s\cos t + \sin t\right)\right]_0^\pi + \left[\frac{e^{-st}}{s^2+1}\left(-s\sin t - \cos t\right)\right]_\pi^\infty$$

$$= \frac{e^{-s\pi}(s)}{s^2+1} + \frac{s}{s^2+1} - \frac{e^{-s\pi}}{s^2+1}$$

$$= \frac{1}{s^2+1}\left[s + e^{-\pi s}(s-1)\right]$$

Result: <u>change of scale property</u>

If $L[f(t)] = F(s)$, then

$$L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right),\ a>0$$

proof:

W.K.T., $L[f(t)] = \int_0^\infty e^{-st} f(at)\,dt$

put $at = x$ when $t=0, x=0$

$a\,dt = dx$ $t=\infty, x=\infty$

$$= \int_0^\infty e^{-s(x/a)} f(x)\,\frac{dx}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{(-s/a)x} f(x)\,dx$$

$$= \frac{1}{a}\int_0^\infty e^{-(s/a)t} f(t)\,dt$$

[illegible]

$= \frac{1}{a} F\left(\frac{s}{a}\right)$

$$\boxed{\therefore L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)}$$

Theorem

If $L[f(t)] = F(s)$, then $L[t f(t)] = -\frac{d}{ds} F(s)$

Proof: $L[f(t)] = F(s)$

$F(s) = L[f(t)]$

Taking derivative on both sides w.r.t. s, we get

$$\frac{d}{ds} F(s) = \frac{d}{ds}[F(s)]$$

$$= \frac{d}{ds}\left[\int_0^\infty e^{-st} f(t)\,dt\right]$$

$$= \int_0^\infty \frac{\partial}{\partial s}\left[e^{-st} f(t)\right] dt$$

$$= \int_0^\infty -t\, e^{-st} f(t)\,dt$$

$$= -\int_0^\infty e^{-st}\left[t f(t)\right] dt$$

$$\frac{d}{ds}[F(s)] = -L[t f(t)]$$

$$\therefore L[t f(t)] = -\frac{d}{ds} F(s)$$

(or)

$$= -F'(s)$$

where $F(s) = L[f(t)]$

Corollary

We have

$$L[t f(t)] = -\frac{d}{ds} F(s)$$

$$\therefore L[t^2 f(t)] = L[t \cdot t f(t)]$$

$$= -\frac{d}{ds} L[t f(t)]$$

$$= -\frac{d}{ds}\left(-\frac{d}{ds} L[f(t)]\right)$$

$$= (-1)^2 \frac{d^2}{ds^2} F(s)$$

Similarly we can easily show that

$$L[t^3 f(t)] = (-1)^3 \frac{d^3}{ds^3} F(s)$$

In general, we have

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$$

Solved problems using

$$L[t f(t)] = -\frac{d}{ds} F(s)$$

Ex 1: Evaluate $L[t \sin 2t]$.

Soln: W.k.t.

$$L[t f(t)] = -\frac{d}{ds} F(s),$$

where $F(s) = L[f(t)]$

Here $f(t) = \sin 2t$

$$F(s) = L[f(t)] = L[\sin 2t] = \frac{2}{s^2+4}$$

$$\therefore F(s) = \frac{2}{s^2+4}$$

$$\therefore L[t \sin 2t] = \frac{-d}{ds}\left(\frac{2}{s^2+4}\right)$$

$$= \left[\frac{0 - 2 \cdot 2s}{(s^2+4)^2}\right]$$

$$= \frac{4s}{(s^2+4)^2}$$

Example 2

Find $L[t^2 e^{-3t}]$.

Solu:-

Wkt, $L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2} F(s)$, where $F(s) = L[f(t)]$

Here, $f(t) = e^{-3t}$

$$L[t^2 e^{-3t}] = (-1)^2 \frac{d^2}{ds^2} L[e^{-3t}]$$

$$= \frac{d^2}{ds^2} \cdot \frac{1}{s+3}$$

$$= \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{1}{s+3}\right)\right]$$

$$= \frac{d}{ds}\left[\frac{0-1}{(s+3)^2}\right]$$

$$= \frac{0+2(s+3)}{(s+3)^4}$$

$$= \frac{2}{(s+3)^3}$$

$$= -\frac{d}{ds} L(\sin \omega t) = \ldots$$

$$= \frac{2\omega s}{(s^2+\omega^2)^2}$$

$f'(t) = \omega t \cos \omega t + \sin \omega t$

$$\therefore L(\omega t \cos \omega t + \sin \omega t) = \frac{s \cdot 2\omega s}{(s^2+\omega^2)^2}$$

$$= \frac{2\omega s^2}{(s^2+\omega^2)^2}$$

Theorem:

If $L[f(t)] = F(s)$ & if $\frac{f(t)}{t}$ has a limit as $t \to 0$ then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$

Proof: Given

$$F(s) = L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$\therefore \int_s^\infty F(s)\,ds = \int_s^\infty \int_0^\infty e^{-st} f(t)\,dt\,ds$$

$$= \int_s^\infty ds \int_0^\infty e^{-st} f(t)\,dt$$

($\because$ s and t are independent variables and hence the order of Integration in the double integral can be interchanged)

$$\therefore \int_s^\infty F(s)\,ds = \int_0^\infty dt \int_s^\infty e^{-st} f(t)\,ds$$

$$= \int_0^\infty f(t)\,dt \int_s^\infty e^{-st}\,ds$$

$$= \int_0^\infty f(t)\,dt \left[\frac{e^{-st}}{-t}\right]_s^\infty$$

$$= \int_0^\infty \frac{e^{-st}}{t} f(t)\,dt$$

$$= \int_0^\infty e^{-st} \left[\frac{f(t)}{t}\right] dt = L\left[\frac{f(t)}{t}\right]$$

$$\therefore L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds.$$

$F(s) = L[f(t)]$

Solved problems using

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$$

Example 1:

Find $L\left[\frac{\sin at}{t}\right]$. Hence show that $\int_0^\infty \frac{\sin t}{t}\,dt = \pi/2$

soln: Here $\lim_{t\to 0}\left[\frac{\sin at}{t}\right] = a$ finite value

$$\therefore L\left[\frac{\sin at}{t}\right] = \int_s^\infty L[\sin at]\,ds$$

$$= \int_s^\infty \frac{a}{s^2+a^2}\,ds = a\int_s^\infty \frac{1}{s^2+a^2}\,ds$$

$$= a\left[\frac{1}{a}\tan^{-1}(s/a)\right]_s^\infty$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s/a)$$

$$= \pi/2 - \tan^{-1}(s/a)$$

$$= \cot^{-1}(s/a)$$

Deduction: We have $L\left[\frac{\sin at}{t}\right] = \pi/2 - \tan^{-1}\ldots$

Laplace transform of Derivatives

Theorem 1:

$$L[f'(t)] = sL[f(t)] - f(0)$$

proof:

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \int_0^\infty e^{-st}\, d[f(t)]$$

$$= [e^{-st} f(t)]_0^\infty - \int_0^\infty f(t)\, d(e^{-st})$$

$$= 0 - e^0 f(0) - \int_0^\infty f(t)\, e^{-st}(-s)\,dt$$

$$= -f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= sL[f(t)] - f(0)$$

similarly,

$$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$$

$$L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - sf'(0) - f''(0)$$

& so on In general,

$$L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{(n-}$$

Laplace transform of integrals

Theorem:

$$L\left[\int_0^t f(x)\,dx\right] = \frac{1}{s} L[f(t)]$$

proof:

Let $\int_0^t f(x)\,dx = F(t)$

$\therefore F'(t) = f(t)$ &

$$F(0) = \int_0^0 f(x)\,dx = 0$$

$$\therefore L[F'(t)] = sL[F(t)] - F(0)$$

$$= sL[F(t)].$$

i.e., $L[f(t)] = sL\left[\int_0^t f(x)\,dx\right]$

i.e., $L\left[\int_0^t f(x)\,dx\right] = \frac{1}{s} L[f(t)]$ $(\because F'(t) = f(t))$

Solved problems under Laplace Transform of integrals:

Example 1:

Find the Laplace transform of

$$e^{-t}\int_0^t \cos t\,dt$$

soln: $L\left[\int_0^t t\cos t\,dt\right] = \frac{1}{s} L(t\cos t)$

$$= \frac{1}{s}\left[-\frac{d}{ds}\, L(\cos t)\right] = \frac{1}{s}\left[-\frac{d}{ds}\right.$$

$$= \frac{1}{s}\left[-\frac{d}{ds}\right.$$

$$= \frac{1}{s}\left[-\frac{(s^2+1) - s \cdot 2s}{(s^2+1)^2}\right]$$

$$= \frac{s^2-1}{s(s^2+1)^2}$$

$$\therefore L\left[e^{-t}\int_0^t t\cos t\, dt\right]$$

$$= \left[\frac{s^2+1}{s(s^2+1)^2}\right]_{s \to s+1}$$

$$= \frac{(s+1)^2-1}{(s+1)\left[(s+1)^2+1\right]^2}$$

$$= \frac{s^2+2s}{(s+1)(s^2+2s+2)^2}$$

(c) $L[e^{-\pi t}\delta(t-a)] = L[\delta(t-a)]_{s\to s+\pi}$

$$= (e^{-at})_{s\to s+\pi}$$

$$= e^{-a(s+\pi)}$$

Periodic Functions

Def: A function $f(t)$ is said to have a period T or to be periodic with period T if for all t, $f(t+T) = f(t)$ where T is a positive constant. The least value of $T>0$ is called the period of $f(t)$.

Eg 1: consider $f(t) = \sin t$

$$f(t+2\pi) = \sin(t+2\pi) = \sin t.$$

i.e., $f(t) = f(t+2\pi) = \sin t$

$\therefore$ $\sin t$ is a periodic function with period 2π.

Eg 2: $\tan t$ is a periodic function with period π since,

$$\tan(t+\pi) = \tan t.$$

Laplace transforms of periodic functions

Let $f(t)$ be a periodic function with period a.

ie, $f(t) = f(t+a) = f(t+2a) = f(t+3a) = \cdots$

Now, $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^a e^{-st} f(t)\,dt + \int_a^{2a} e^{-st} f(t)\,dt$$

$$+ \int_{2a}^{3a} e^{-st} f(t)\,dt + \int_{3a}^{4a} e^{-st} f(t)\,dt + \cdots$$

put in the second integral $t = T + a$; $\Rightarrow dt = dT$

in the third integral $t = T + 2a$; $\Rightarrow dt = dT$

in the fourth integral $t = T + 3a$; $\Rightarrow dt = dT$

When $t = a,\ T = 0$; $t = 2a,\ T = a$ } in 1st integral

When $t = 2a,\ T = 0$; $t = 3a,\ T = a$ } in 2nd integral

When $t = 3a,\ T = 0$; $t = 4a,\ T = a$ } in 3rd integral

$L[f(t)]$

$$= \int_0^a e^{-st} f(t)\,dt + e^{-as}\int_0^a e^{-sT} f(T+a)\,dT + e^{-2as}\int_0^a e^{-sT} f(T+2a)\,dT + \dots$$

$$= \int_0^a e^{-st} f(t)\,dt + e^{-as}\int_0^a e^{-st} f(t+a)\,dt + e^{-2as}\int_0^a e^{-st} f(t+2a)\,dt + \dots$$

$$= [1 + e^{-as} + (e^{-as})^2 + \dots] \int_0^a e^{-st} f(t)\,dt$$

$[\because f(t) = f(t+a) = f(t+2a) = \dots]$

$$= (1 - e^{-as})^{-1} \int_0^a e^{-st} f(t)\,dt$$

$[\because (1-x)^{-1} = 1 + x + x^2 + \dots]$

$$L[f(t)] = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)\,dt$$

Ex: Find the Laplace transform of the rectangular wave given by

$$f(t) = \begin{cases} 1, & 0 < t < b \\ -1, & b < t < 2b \end{cases} \quad \text{with } f(t+2b) = f(t)$$

Soln: Given $f(t) = \begin{cases} 1, & 0 < t < b \\ -1, & b < t < 2b \end{cases}$

This function is periodic in the interval $(0, 2b)$ with period $2b$

$$\therefore L[f(t)] = \frac{1}{1-e^{-2bs}} \int_0^{2b} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2bs}} \left[\int_0^b e^{-st} f(t)\,dt + \int_b^{2b} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-2bs}} \left[\int_0^b e^{-st}(1)\,dt + \int_b^{2b} e^{-st}(-1)\,dt\right]$$

$$= \frac{1}{1-e^{-2bs}} \left\{\left[\frac{e^{-st}}{-s}\right]_0^b - \left[\frac{e^{-st}}{-s}\right]_b^{2b}\right\}$$

$$= \frac{1}{1-e^{-2bs}} \left[\frac{e^{-bs}}{-s} + \frac{1}{s} + \frac{e^{-2bs}}{s} - \dots\right.$$

$$= \frac{1}{s}\left[\frac{1-2e^{-sb}+e^{-2sb}}{1-e^{-2sb}}\right]$$

$$= \frac{1}{s}\,\frac{(1-e^{-sb})^2}{(1+e^{-sb})(1-e^{-sb})}$$

$$= \frac{1}{s}\cdot\frac{1-e^{-sb}}{1+e^{-sb}} = \frac{1}{s}\cdot\frac{1-e^{-\frac{sb}{2}}\cdot e^{-\frac{sb}{2}}}{1+e^{-sb/2}\cdot e^{-sb/2}}$$

$$= \frac{1}{s}\cdot\frac{e^{+sb/2}-e^{-sb/2}}{e^{sb/2}+e^{-sb/2}}$$

$$= \frac{1}{s}\tanh\left(\frac{sb}{2}\right) \qquad \left(\because \tanh x = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\right)$$

Ex: 2 Find the Laplace transform of the Half-wave rectifier function $f(t) = \begin{cases} \sin\omega t, & 0 < t < \pi/\omega \\ 0, & \pi/\omega < t < 2\pi/\omega \end{cases}$ with $f\left(t+\frac{2\pi}{\omega}\right) = f(t)$

Soln:-

Given $f(t) = \begin{cases} \sin\omega t, & 0 < t < \pi/\omega \\ 0, & \pi/\omega < t < 2\pi/\omega \end{cases}$

This is a periodic function with period ~~2b~~ $2\pi/\omega$ in the interval $(0, 2\pi/\omega)$

$$\therefore L[f(t)] = \frac{1}{1-e^{-2bs}}\int_0^{2b} e^{-st} f(t)\,dt.$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\int_0^{\pi/\omega} e^{-st} f(t)\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}\,0\,dt\right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-st}(-s\sin\omega t - \omega\cos\omega t)}{s^2+\omega^2}\right]_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-2\pi/\omega}(\omega)+(\omega)}{s^2+\omega^2}\right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\cdot\frac{\omega\left(1+e^{-2\pi/\omega}\right)}{s^2+\omega^2} \qquad \left[\because \begin{matrix}\cos\pi = -1 \\ \sin\pi = 0\end{matrix}\right]$$

$$= \frac{\omega}{\left(1-e^{-2\pi/\omega}\right)(x^2+\omega^2)}$$

Ex: 2 Find the Laplace transform of the function $f(t) = \begin{cases} t, & 0 < t < b \\ 2b-t, & b < t < 2b \end{cases}$ with $f(t+2b) = f(t)$

Soln:

The given function is a periodic function with period $2b$.

$$\therefore L[f(t)] = \frac{1}{1-e^{-2bs}}\int_0^{2b} e^{-st} f(t)\,d$$

$$= \frac{1}{1-e^{-2bs}}\left[\int_0^{b} e^{-st} f(t)\,dt + \int_b^{2b} e^{-st} f\right.$$

$$= \frac{1}{1-e^{-2bs}}\left[\int_0^b e^{-st}\, t\,dt + \int_b^{2b} e^{-st}(2b-1)\,dt\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[s\left(\frac{e^{-st}}{-s}\right) - 1\left(\frac{e^{-st}}{s^2}\right)\right]_0^b$$

$$+ \left[(2b-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right)\right]_b^{2b}$$

$$= \frac{1}{1-e^{-2bs}}\left[-\frac{be^{-sb}}{s} - \frac{e^{-sb}}{s^2} + \frac{1}{s^2} + \frac{e^{-2bs}}{s^2} + \frac{b}{s}e^{-sa} - \frac{e^{-bs}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\frac{1-2e^{-sb}+e^{-2bs}}{s^2}\right]$$

$$= \frac{(1-e^{-bs})^2}{s^2(1+e^{-bs})(1-e^{-bs})}$$

$$= \frac{1-e^{-bs}}{s^2(1+e^{-bs})} = \frac{1}{s^2}\cdot\frac{1\cdot e^{-bs/2}\cdot e^{-b}}{1+e^{-bs/2}\cdot e^{-b}}$$

$$= \frac{1}{s^2}\cdot\frac{e^{bs/2}-e^{-bs/2}}{e^{bs/2}+e^{-bs/2}}$$

$$= \frac{1}{s^2}\cdot\frac{e^{bs/2}-e^{-bs/2}}{e^{bs/2}+e^{-bs/2}}$$

$$= \frac{1}{s^2}\tanh(bs/2).$$

The inverse Laplace transform.

Def: If the Laplace transform of a function $f(t)$ is $F(s)$ i.e., $L[f(t)] = F(s)$ then $f(t)$ is called an inverse Laplace transform of $F(s)$ and is denoted by.

$$f(t) = L^{-1}[F(s)].$$

Here L^{-1} is called the inverse Laplace transform operator. Thus if

$$L[e^{at}] = \frac{1}{s-a}$$

then $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

Similarly we can obtain some standard inverse Laplace Transform as follows.

$L[e^{at}] = 1/s-a$

1. $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

$L[e^{-at}] = 1/s+a$

2. $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$ $\qquad L[\sin at] = \frac{a}{s^2+a^2}$

3. $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$

4 $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$ $\quad L[\cos at] = \frac{s}{s^2+a^2}$

5 $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{\sinh at}{a}$ $\quad L[\sinh at] = \frac{a}{s^2-a^2}$

6 $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$ $\quad L[\cosh at] = \frac{s}{s^2-a^2}$

7 $L^{-1}[1/s] = 1$ $\quad L[1] = 1/s$

8 $L^{-1}[1/s^2] = t$ $\quad L[t] = 1/s^2$

9 $L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$ $\quad L[t^n] = \frac{n!}{s^{n+1}}$

10 $L^{-1}\left[\frac{1}{(s-a)^2}\right] = te^{at}$ $\quad L[te^{at}] = \frac{1}{(s-a)^2}$

Linear property

If $F_1(s)$ & $F_2(s)$ are laplace transform of $f_1(t)$ & $f_2(t)$ respectively, then

$$L^{-1}[c_1F_1(s) + c_2F_2(s)] = c_1L^{-1}[F_1(s)] + c_2L^{-1}[F_2(s)]$$

where c_1 & c_2 are constants

proof WKT,

$$L[c_1f_1(t) + c_2f_2(t)] = c_1L[f_1(t)] + c_2L[f_2(t)]$$
$$= c_1F_1(s) + c_2F_2(s)$$

$[\because L[f_1(t)] = F_1(s)$ & $L[f_2(t)] = F_2(s)]$

$$c_1f_1(t) + c_2f_2(t) = L^{-1}[c_1F_1(s) + c_2F_2(s)]$$

i.e. $c_1L^{-1}[F_1(s)] + c_2L^{-1}[F_2(s)]$
$= L^{-1}[c_1F_1(s) + c_2F_2(s)]$

problems based on linear property

Example 1

Find $L^{-1}\left[\frac{1}{s-3} + \frac{1}{s} + \frac{s}{s^2-4}\right]$

solu:

$$L^{-1}\left[\frac{1}{s-3} + \frac{1}{s} + \frac{s}{s^2-4}\right] = L^{-1}\left[\frac{1}{s-3}\right] + L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{s}{s^2-4}\right]$$

$= e^{3t} + 1 + \cosh 2t$

$= e^{3t} + \cosh 2t + 1$

Example 2 Find $L^{-1}\left[\frac{1}{s^2} + \frac{1}{s+4} + \frac{1}{s^2+4} + \frac{s}{s^2-9}\right]$

solu:

$$L^{-1}\left[\frac{1}{s^2} + \frac{1}{s+4} + \frac{1}{s^2+4} + \frac{s}{s^2-9}\right]$$

$$= L^{-1}\left[\frac{1}{s^2}\right] + L^{-1}\left[\frac{1}{s+4}\right] + L^{-1}\left[\frac{1}{s^2+2^2}\right] + L^{-1}\left[\frac{s}{s^2-3^2}\right]$$

$= t + e^{-4t} + \frac{\sin 2t}{2} + \cosh 3t$

Example-3

Find $L^{-1}\left[\frac{1}{s-3/2} + \frac{s}{s^2-1}\right]$

solu:

$$L^{-1}\left[\frac{1}{s-3/2} + \frac{s}{s^2-1}\right] = L^{-1}\left[\frac{1}{s-3/2}\right] + L^{-1}\left[\frac{s}{s^2-1}\right]$$

$= e^{3/2\, t} + \cosh t$

Example-4

Find $L^{-1}\left[\frac{s}{s^2+49} + \frac{1}{s+\sqrt{7}}\right]$

solu:

$$L^{-1}\left[\frac{s}{s^2+49} + \frac{1}{s+\sqrt{7}}\right] = L^{-1}\left[\frac{s}{s^2+7^2}\right] + L^{-1}\left[\frac{1}{s+\ldots}\right]$$

$= \cos 4t + e^{-\sqrt{7}t}$

Example - 5

Find $L^{-1}\left[\frac{2s}{s^2-25} + \frac{1}{s^3}\right]$

solu:

$L^{-1}\left[\frac{2s}{s^2-25} + \frac{1}{s^3}\right] = 2L^{-1}\left[\frac{s}{s^2-5^2}\right] + L^{-1}\left[\frac{1}{s^3}\right]$

$= 2\cosh 5t + \frac{1}{2} t^2$

Example - 6

Find $L^{-1}\left[\frac{s}{s^2-225} + \frac{1}{s-\sqrt{7}} + \frac{1}{s^2+16}\right]$

solu:-

$L^{-1}\left[\frac{s}{s^2-225} + \frac{1}{s-\sqrt{7}} + \frac{1}{s^2+16}\right]$

$= L^{-1}\left[\frac{s}{s^2-225}\right] + L^{-1}\left[\frac{1}{s-\sqrt{7}}\right] + L^{-1}\left[\frac{1}{s^2+16}\right]$

$= \cosh 15t + e^{\sqrt{7}t} + \frac{1}{4}\sin 4t$

First shifting property

Result 1: WKT

if $L[f(t)] = F(s)$, then

Hence $L[e^{-at} f(t)] = F(s+a)$

$L^{-1}[F(s+a)] = e^{-at} f(t)$

$= e^{-at} L^{-1}[F(s)]$

$[\because f(t) = L^{-1}[F(s)]]$

Thus if we replace s by s+a in F(s), the inverse Laplace transform of the resulting function F(s+a) is e^{at} times the inverse laplace transform of F

If we know that $L^{-1}[F(s)] = f(t)$, we can immediately find the transform of F(s+a) by multiplying f(t) by e^{-at}

problems based on First shifting property

Example 1:

Find $L^{-1}\left[\frac{1}{(s+1)^2}\right]$

solu:

$L^{-1}\left[\frac{1}{(s+1)^2}\right] = e^{-t} L^{-1}\left[\frac{1}{s^2}\right] = e^{-t}\, t$

Example 2:

Find $L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$

solu:

$L^{-1}\left[\frac{1}{(s+1)^2+1}\right] = e^{-t} L^{-1}\left[\frac{1}{s^2+1}\right] = e^{-t}\sin t$

Example 3

Find $L^{-1}\left[\frac{s}{(s+2)^2+1}\right]$

solu:

$L^{-1}\left[\frac{s}{(s+2)^2+1}\right] = L^{-1}\left[\frac{s+2-2}{(s+2)^2+1}\right]$

$= L^{-1}\left[\frac{s+2}{(s+2)^2+1}\right] - 2L^{-1}\left[\frac{1}{(s+2)^2+1}\right]$

$= e^{-2t} L^{-1}\left[\frac{s}{s^2+1}\right] - 2e^{-2t} L^{-1}\left[\frac{1}{s^2 \ldots}\right]$

$= e^{-2t}\cdot\cos t - 2e^{-2t}\cdot\sin t$

$= e^{-2t}(\cos t - 2\sin t)$

Example 4:

Find $L^{-1}\left[\frac{s-3}{(s-3)^2+4}\right]$

solu:

$L^{-1}\left[\frac{s-3}{(s-3)^2+4}\right] = e^{3t} L^{-1}\left[\frac{s}{s^2+4}\right] = e^{3t}\cdot \cos \ldots$

eg 12 Find $L^{-1}\left[\frac{s+7}{(s+7)^2+49}\right]$

solu:

$$L^{-1}\left[\frac{s+7}{(s+7)^2+49}\right] = e^{-7t}\, L^{-1}\left[\frac{s}{s^2+49}\right]$$

$$= e^{-7t}\cos 7t$$

change of scale property

If $L[f(t)] = F(s)$, then

$$L^{-1}[F(as)] = \frac{1}{a} f(t/a),\ a>0$$

proof

$$F(s) = L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$F(as) = \int_0^\infty e^{-ast} f(t)\,dt$$

put $at = t_1$, when $t=0$, $t_1 = 0$

$dt = \frac{dt_1}{a}$ $\quad t=\infty$, $t_1=\infty$

$$= \int_0^\infty e^{-st_1} f\left(\frac{t_1}{a}\right)\frac{dt_1}{a} = \frac{1}{a}\int_0^\infty e^{-st_1} f\left(\frac{t_1}{a}\right)dt_1$$

$$= \frac{1}{a}\int_0^\infty e^{-st} f\left(\frac{t}{a}\right)dt$$

$$\left[\because \int_a^b f(t)\,dt = \int_a^b f(t_1)\,dt_1\right]$$

$$= \frac{1}{a} L\left[f(t/a)\right]$$

$$\therefore L^{-1}[F(as)] = \frac{1}{a} f(t/a),\ a>0$$

problems based on change of scale property

eg 1 If $L^{-1}\left[\frac{s^2-1}{(s^2+1)^2}\right] = t\cos t$, then find $L^{-1}\left[\frac{9s^2-1}{(9s^2+1)^2}\right]$

solu: $L^{-1}\left[\frac{s^2-1}{(s^2+1)^2}\right] = t\cos t$

Writing as for s

$$L^{-1}\left[\frac{a^2s^2-1}{(a^2s^2+1)^2}\right] = \frac{1}{a}\cdot\frac{t}{a}\cos(t/a)$$

putting $a=3$

$$L^{-1}\left[\frac{9s^2-1}{(9s^2+1)^2}\right] = \frac{1}{3}\cdot\frac{t}{3}\cos(t/3)$$

$$= t/9\,\cos(t/3)$$

eg 2 Find $L^{-1}\left[\frac{s}{2s^2-8}\right]$

solu: we know

$$L^{-1}\left[\frac{s}{s^2-4^2}\right] = \cosh 4t$$

putting 2s for s

$$L^{-1}\left[\frac{2s}{(2s)^2-4^2}\right] = \frac{1}{2}\cosh(4t/2)$$

$$L^{-1}\left[\frac{2s}{4s^2-16}\right] = \frac{1}{2}\cosh 2t$$

i.e., $L^{-1}\left[\frac{s}{2s^2-8}\right] = \frac{1}{2}\cosh 2t$

eg 3 Find $L^{-1}\left[\frac{s}{s^2a^2+b^2}\right]$

solu:

$$\frac{s}{s^2a^2+b^2} = \frac{1}{a}\cdot\frac{as}{s^2a^2+b^2} = \frac{1}{a} F(a$$

where $F(s) = s/s^2+b^2$

$$\therefore L^{-1}\left[\frac{s}{s^2a^2+b^2}\right] = \frac{1}{a} L^{-1}\left[\frac{sa}{s^2a^2+b^2}\right]$$

$$= \frac{1}{a} L^{-1}[F(as)] = \frac{1}{a}\cdot\frac{1}{a}\cdot f(t/a$$

$$\therefore L^{-1}\left[\frac{2(s+1)}{(s^2+2s+2)^2}\right] = -t\, L^{-1}\left[\frac{-1}{s^2+2s+2}\right]$$

$$= t \cdot L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= t \cdot e^{-t} L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= t \cdot e^{-t} \sin t$$

Result: If $L[f(t)] = sF(s)$ and $\phi(t)$ is a function such that $L[\phi(t)] = F(s)$ & $\phi(0)=0$, then $f(t) = \phi'(t)$

proof: WKT,

$$L[\phi'(t)] = sL[\phi(t)] - \phi(0) = sF(s)$$

$(\because \phi(0)=0)$

$$= L[f(t)]$$

i.e., $L[\phi'(t)] = L[f(t)]$

$$\therefore \phi'(t) = f(t)$$

From this result, we get,

$$L^{-1}[sF(s)] = f(t) = \phi'(t) = \frac{d\phi(t)}{dt}$$

$$= \frac{d}{dt} L^{-1}[F(s)]$$

$[\because L[\phi(t)] = F(s)]$.

provided, $L^{-1}[F(s)] = 0$ as $t \to 0$

$$L^{-1}[sF(s)] = \frac{d}{dt} L^{-1}[F(s)].$$

The above result can be used only when $\lim_{t\to 0} L^{-1}[F(s)] = 0$

Problems based on $L^{-1}[sF(s)] = \frac{d}{dt} L^{-1}[F(s)]$.

Eg 1: Find $L^{-1}\left[\frac{s}{(s+2)^2+4}\right]$

soln:-

$$L^{-1}\left[\frac{s}{(s+2)^2+4}\right] = L^{-1}\left[s \cdot \frac{1}{(s+2)^2+4}\right]$$

$$= \frac{d}{dt} L^{-1}\left[\frac{1}{(s+2)^2+4}\right]$$

(using the above result).

$$= \frac{d}{dt} e^{-2t} L^{-1}\left[\frac{1}{s^2+2^2}\right]$$

$$= \frac{d}{dt}\left(e^{-2t} \cdot \frac{1}{2}\sin 2t\right)$$

$$= \frac{1}{2}\left[2e^{-2t}\cos 2t + \sin 2t\, e^{-2t}(-2)\right]$$

$$= e^{-2t}(\cos 2t - \sin 2t)$$

Another method.

$$L^{-1}\left[\frac{s}{(s+2)^2+4}\right] = L^{-1}\left[\frac{s+2-2}{(s+2)^2+4}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+4}\right] - 2L^{-1}\left[\frac{1}{(s+2)^2+4}\right]$$

$$= e^{-2t} L^{-1}\left[\frac{s}{s^2+2^2}\right] - 2e^{-2t} L^{-1}\left[\frac{1}{s^2+2^2}\right]$$

$$= e^{-2t}\cos 2t - 2e^{-2t} \cdot \frac{1}{2}\sin 2t$$

$$= e^{-2t}(\cos 2t - \sin 2t)$$

Eg:2 Find $L^{-1}\left[\frac{s}{(s+2)^2}\right]$

soln:-

$$L^{-1}\left[\frac{s}{(s+2)^2}\right] = L^{-1}\left[s \cdot \frac{1}{(s+2)^2}\right]$$

$$= \frac{d}{dt} L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= \frac{d}{dt} \cdot e^{-2t} L^{-1}\left[\frac{1}{s^2}\right] = \frac{d}{dt}(e^{-2t} \cdot t)$$

$$= e^{-2t} + t\, e^{-2t}(-2) = e^{-2t} \cdot (1-2t)$$

Another method:

$$L^{-1}\left[\frac{s}{(s+2)^2}\right] = L^{-1}\left[\frac{s+2-2}{(s+2)^2}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2}\right] - L^{-1}\left[\frac{2}{(s+2)^2}\right]$$

$$= L^{-1}\left[\frac{1}{s+2}\right] - 2e^{-2t} L^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^{-2t} - 2e^{-2t} \cdot t$$

$$= e^{-2t}(1-2t).$$

Eg: 3 Find $L^{-1}\left[\frac{s^2}{(s-2)^3}\right]$

solu:-

$$L^{-1}\left[\frac{s^2}{(s-2)^3}\right] = L^{-1}\left[s \cdot \frac{s}{(s-2)^3}\right] = \frac{d}{dt}\left\{L^{-1}\left(\frac{1}{(s-2)^3}\right)\right\}$$

$$= \frac{d}{dt} \cdot \frac{d}{dt} L^{-1}\left[\frac{1}{(s-2)^3}\right]$$

$$= \frac{d^2}{dt^2} \cdot e^{2t} L^{-1}\left[\frac{1}{s^3}\right]$$

$$= \frac{d^2}{dt^2}\left(\frac{e^{2t} \cdot t^2}{2}\right) = \frac{d}{dt}\left((t^2+1)e^{2t}\right)$$

$$= (2t^2 + 4t + 1)\, e^{2t}.$$

Eg: 4 Find $L^{-1}\left[\frac{s-3}{s^2+4s+13}\right]$

solu:

$$= e^{at}\frac{t^3}{24}[4+at]$$

Theorem:

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]\,dt$$

proof: WKT,

ie, $L\left[\int_0^t f(x)\,dx\right] = \frac{1}{s}L[f(t)]$

$$\therefore \int_0^t f(x)\,dx = L^{-1}\left\{\frac{1}{s}L[f(t)]\right\}$$

ie, $L^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t f(t)\,dt$ $\quad [\because F(s) = L[f(t)]]$

$$= \int_0^t L^{-1}[F(s)]\,dt \quad [\because f(t) = L^{-1}[F(s)]]$$

$$\therefore L^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t L^{-1}[F(s)]\,dt.$$

problems based on $L^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t L^{-1}[F(s)]\,dt$

Eg: 1 Find $L^{-1}\left[\frac{1}{s(s+3)}\right]$

solu:-

$$L^{-1}\left[\frac{1}{s(s+3)}\right] = L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s+3}\right] = \int_0^t L^{-1}\left[\frac{1}{s+3}\right]dt$$

$$= \int_0^t e^{-3t}\,dt = \left[\frac{e^{-3t}}{-3}\right]_0^t$$

$$= \frac{e^{-3t}}{-3} + \frac{1}{3} = \frac{1-e^{-3t}}{3}.$$

Eg: 2 Find $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$

solu:-

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s^2+a^2}\right]$$

TO Find inverse Laplace transforms of
LOG [F(S)]

Eg:1 Find $L^{-1}\left[\log\left(\frac{1+s}{s^2}\right)\right]$

solu:-

let $L^{-1}\left[\log\left(\frac{1+s}{s^2}\right)\right] = f(t)$

$$\therefore L[f(t)] = \log \frac{1+s}{s^2}$$

$$L[t f(t)] = -\frac{d}{ds}\left[\log\left(\frac{1+s}{s^2}\right)\right]$$

$$= \frac{-d}{ds}\left[\log(1+s) - \log(s^2)\right]$$

$$= -\frac{1}{1+s} + \frac{1}{s^2}\cdot 2s$$

$$L[t f(t)] = \frac{2}{s} - \frac{1}{s+1}$$

$$\therefore t f(t) = L^{-1}\left[\frac{2}{s} - \frac{1}{s+1}\right]$$

$$= 2L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right]$$

$$= 2\cdot 1 - e^{-t}$$

$$\therefore f(t) = \frac{2-e^{-t}}{t}$$

$$\therefore L^{-1}\left[\log\left(\frac{1+s}{s^2}\right)\right] = \frac{2-e^{-t}}{t}$$

Eg:2 Find $L^{-1}\left[\log\left(1+\frac{\omega^2}{s^2}\right)\right]$

solu:- Let, $L^{-1}\left[\log\left(1+\frac{\omega^2}{s^2}\right)\right] = f(t)$

$$\therefore L[f(t)] = \log\left(1+\frac{\omega^2}{s^2}\right) = \log\left(\frac{s^2+\omega^2}{s^2}\right)$$

$$= \log(s^2+\omega^2) - \log(s^2)$$

$$L[t f(t)] = \frac{-d}{ds}\left[\log(s^2+\omega^2) - \log(s^2)\right]$$

Eg 5 Find $L^{-1}\left[\log \frac{s-a}{s^2+a^2}\right]$

solu:

Let $L^{-1}\left[\log \frac{s-a}{s^2+a^2}\right] = f(t)$

$\therefore L[f(t)] = \log\left[\frac{s-a}{s^2+a^2}\right]$

$= \log(s-a) - \log(s^2+a^2)$

$L[t f(t)] = -\frac{d}{ds}\left[\log(s-a) - \log(s^2+a^2)\right]$

$= -\frac{1}{s-a} + \frac{2s}{s^2+a^2}$

$\therefore t f(t) = L^{-1}\left[\frac{2s}{s^2+a^2} - \frac{1}{s-a}\right]$

$= L^{-1}\left[\frac{2s}{s^2+a^2}\right] - L^{-1}\left[\frac{1}{s-a}\right]$

$= 2\cos at - e^{at}$

$f(t) = \frac{2\cos at - e^{at}}{t}$

$L^{-1}\left[\log \frac{s-a}{s^2+a^2}\right] = \frac{2\cos at - e^{at}}{t}$

Eg 6 Find $L^{-1}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)$

solu:

Let $L^{-1}\log\left(\frac{s^2+a^2}{s^2+b^2}\right) = f(t)$

$\therefore L[f(t)] = \log\left(\frac{s^2+a^2}{s^2+b^2}\right)$

$= \log(s^2+a^2) - \log(s^2+b^2)$

$L[t f(t)] = -\frac{d}{ds}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]$

$= -\frac{1}{s^2+a^2}\cdot 2s + \frac{1}{s^2+b^2}\cdot 2s$

$= \frac{2s}{s^2+b^2} - \frac{2s}{s^2+a^2}$

$t f(t) = L^{-1}\left[\frac{2s}{s^2+b^2} - \frac{2s}{s^2+a^2}\right]$

$= 2L^{-1}\left[\frac{s}{s^2+b^2}\right] - 2L^{-1}\left[\frac{s}{s^2+a^2}\right]$

$= 2\cos bt - 2\cos at$

$= 2(\cos bt - \cos at)$

$\therefore f(t) = \frac{2(\cos bt - \cos at)}{t}$

$L^{-1}\log\left(\frac{s^2+a^2}{s^2+b^2}\right) = \frac{2(\cos bt - \cos at)}{t}$

Eg 7 Find the Laplace inverse of $s\log\left(\frac{s-1}{s+1}\right)+2$

solu:

Let $L^{-1}\left(s\log\left(\frac{s-1}{s+1}\right)+2\right) = f(t)$

$\therefore L[f(t)] = s\log\left(\frac{s-1}{s+1}\right)+2$

$= s\log(s-1) - s\log(s+1) + 2$

$L[t f(t)] = -\frac{d}{ds}\left[s\log(s-1) - s\log(s+1) + 2\right]$

$= -\left[\frac{s}{s-1} + \log(s-1) - \frac{s}{s+1} - \log(s+1)\right]$

$= \log\left(\frac{s+1}{s-1}\right) - \frac{2s}{s^2-1}$

$t f(t) = L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right] - 2L^{-1}\left[\frac{s}{s^2-1}\right]$

$= L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right] - 2\cosh t \quad -(1)$

To find $L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$

Let $f(t) = L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$;

$L[f(t)] = \log\left(\frac{s+1}{s-1}\right)$

$L[t f(t)] = -\frac{d}{ds}\left[\log(s+1) - \log(s-1)\right]$

$= \frac{1}{s-1} - \frac{1}{s+1} = \frac{2}{s^2-1}$

$$\therefore t\,f(t) = L^{-1}\left[\frac{2}{s^2-1}\right] = 2\sinh t$$

$$f(t) = \frac{2\sinh t}{t} \quad -(2)$$

Substituting (2) in (1) we get,

$$t\,f(t) = \frac{2\sinh t}{t} - 2\cosh t$$

$$f(t) = \frac{2\sinh t}{t^2} - \frac{2\cosh t}{t}$$

$$f(t) = \frac{2(\sinh t - t\cosh t)}{t^2}$$

$$\therefore L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right] = \frac{2(\sinh t - t\cosh t)}{t^2}$$

Eg: 8 Find $L^{-1}\left[s\cdot\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$

Solu:- Let $L^{-1}\left[s\cdot\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = f(t)$

ie., $L[f(t)] = s\cdot\log\left(\frac{s^2+a^2}{s^2+b^2}\right)$

$$\therefore L[t\,f(t)] = -\frac{d}{ds}\left[s\cdot\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= -s\cdot\frac{1}{\frac{s^2+a^2}{s^2+b^2}}\left[\frac{(s^2+b^2)2s-(s^2+a^2)\cdot 2s}{(s^2+b^2)^2}\right] - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)$$

$$= \frac{-2s^2(b^2-a^2)}{(s^2+a^2)(s^2+b^2)} - \log\left(\frac{s^2+a^2}{s^2+b^2}\right) \quad -(1)$$

$$\therefore t\,f(t) = L^{-1}\left[\frac{2(a^2-b^2)s^2}{(s^2+a^2)(s^2+b^2)}\right] - L^{-1}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

ie, $t\,f(t) = 2(a\sin at - b\sin bt) - \frac{2(\cos bt - \cos at)}{t}$

$$\left\{\because L^{-1}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = \frac{2(\cos bt - \cos at)}{t}\right.$$

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \frac{1}{a^2-b^2}\left\{a\sin at - b\sin bt\right.$$

Method of partial fractions

Eg 1 Find $L^{-1}\left[\frac{1}{(s+1)(s+3)}\right]$

solu:- Let, $F(s) = \frac{1}{(s+1)(s+3)}$

Let us split $F(s)$ into partial fraction

$$\frac{1}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

$$\frac{1}{(s+1)(s+3)} = \frac{A(s+3)+B(s+1)}{(s+1)(s+3)}$$

$$A(s+3)+B(s+1)=1$$

putting $s=-1$ $\quad A(-1+3)+B(-1+1)=1$

$$A(2)+0=1$$

$$A=1/2$$

putting $s=-3$

$$A(-3+3)+B(-3+1)=1$$

$$0-2B=1$$

$$B=-1/2$$

$$\therefore \frac{1}{(s+1)(s+3)} = \frac{\frac{1}{2}}{s+1} + \frac{-1/2}{s+3}$$

$$\therefore L^{-1}\left[\frac{1}{(s+1)(s+3)}\right] = \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s+3}\right]$$

$$= \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}$$

$$= 1/2\,(e^{-t} - e^{-3t})$$

Eg 2 Find $L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$

solu:-

Let, $F(s) = \frac{1}{s(s+1)(s+2)}$

Let us split $F(s)$ into partial fractions

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \quad -(1)$$

$$\frac{1}{s(s+1)(s+2)} = \frac{A(s+1)(s+2)+B(s)(s+2)+C(s)(s+1)}{s(s+1)(s+2)}$$

putting,

$$A(s+1)(s+2)+B(s)(s+2)+C(s)(s+1)=1$$

putting $s=0$, $\quad A(0+1)(0+2)+B(0)(0+2)+C(0)(0+1)=1$

$$A(2)+0+0=1$$

$$A=1/2$$

putting $s=-1$, $\quad A(-1+1)(-1+2)+B(-1)(-1+2)+C(-1)(-1+1)=1$

$$0+B(-1)+0=1$$

$$B=-1$$

putting $s=-2$,

$$A(-2+1)(-2+2)+B(-2)(-2+2)+C(-2)(-2+1)=1$$

$$0+0+2C=1$$

$$C=1/2$$

$A=1/2$, $B=-1$, $C=1/2$ $\rightarrow$ (2)

substituting (2) in (1) we get,

$$\therefore \frac{1}{s(s+1)(s+2)} = \frac{\frac{1}{2}}{s} + \frac{-1}{s+1} + \frac{1/2}{s+2}$$

$$\therefore L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = L^{-1}\left[\frac{1/2}{s} + \frac{-1}{s+1} + \frac{1/2}{s+2}\right]$$

$$= 1/2\,L^{-1}[1/s] - L^{-1}\left[\frac{1}{s+1}\right] + 1/2\,L^{-1}\left[\frac{1}{s+2}\right]$$

$$= 1/2 \cdot 1 - e^{-t} + 1/2\,e^{-2t}$$

$$= 1/2\,(e^{-2t} - 2e^{-t} + 1)$$

Eg 3 Find $L^{-1}\left[\frac{1-s}{(s+1)(s^2+4s+13)}\right]$

solu:- Let, $F(s) = \frac{1-s}{(s+1)(s^2+4s+13)}$

Let us split $F(s)$ into partial fraction

$$\frac{1-s}{(s+1)(s^2+4s+13)} = \frac{A}{s+1} + \frac{Bx+C}{s^2+4s+13}$$

$$\frac{1-s}{(s+1)(s^2+4s+13)} = \frac{A(s^2+4s+13)+(Bx+C)(s+1)}{(s+1)(s^2+4s+13)}$$

$$A(s^2+4s+13)+(Bx+C)(s+1) = 1-s$$

putting $s=-1$

$$A((-1)^2+4(-1)+13)+(Bx+C)((-1)+1) = 1-(-1)$$

$$2 = 10A$$

$$A = 1/5$$

Equating coefficient of s^2

$$A+B=0$$

$$B=-A$$

$$B=-1/5$$

Equating constant coefficient

$$13A+C=1$$

$$13(1/5)+C=1$$

$$C = 1-13/5 = -8/5$$

ie, $C=-8/5$

$$\therefore \frac{1-s}{(s+1)(s^2+4s+13)} = \frac{1/5}{s+1} + \frac{-1/5\,s-8/5}{s^2+4s+13}$$

$$L^{-1}\left[\frac{1-s}{(s+1)(s^2+4s+13)}\right] = \frac{1}{5}L^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{5}L^{-1}\left[\frac{s+8}{s^2+4s+13}\right]$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}L^{-1}\left[\frac{s+2+6}{(s+2)^2+9}\right]$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}L^{-1}\left[\frac{s+2}{(s+2)^2+3^2}\right] - \frac{1}{5}L^{-1}\left[\frac{6}{(s+2)^2+3^2}\right]$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}L^{-1}\left[\frac{s}{s^2+3^2}\right] - \frac{6}{5}e^{-2t}L^{-1}\left[\frac{1}{s^2+3^2}\right]$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}\cos 3t - \frac{6}{5}e^{-2t}\frac{\sin 3t}{3}$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}\cos 3t - \frac{2}{5}e^{-2t}\sin 3t$$

Eg:4 Find $L^{-1}\left[\frac{4s^2-3s+5}{(s+1)(s^2-3s+2)}\right]$

Solu:- Let,

$$F(s) = \frac{4s^2-3s+5}{(s+1)(s^2-3s+2)}$$

Let us split $F(s)$ into partial fraction

$$\frac{4s^2-3s+5}{(s+1)(s^2-3s+2)} = \frac{A}{s+1} + \frac{Bx+C}{(s^2-3s+2)}$$

$$\frac{4s^2-3s+5}{(s+1)(s^2-3s+2)} = \frac{A(s^2-3s+2)+(Bx+C)(s+1)}{(s+1)(s^2-3s+2)}$$

$$A(s^2-3s+2)+(Bx+C)(s+1) = 4s^2-3s+5$$

putting $s=-1$,

$$A((-1)^2-3(-1)+2)+(Bx+C)(-1+1) = 4(-1)^2-3(-1)+5$$

$$6A = 12$$

$$A = 2$$

Equating coefficient of s^2,

ie, $4 = A+B$

$$B = 2$$

Equating constant coefficients,

$$5 = 2A+C$$

$$C = 5-2A = 5-4$$

$$\therefore C = 1$$

$$\therefore \frac{4s^2-3s+5}{(s+1)(s^2-3s+2)} = \frac{2}{s+1} + \frac{2x+1}{s^2-3s+2}$$

$$L^{-1}\left[\frac{4s^2-3s+5}{(s+1)(s^2-3s+2)}\right] = L^{-1}\left[\frac{2}{s+1}\right] + L^{-1}\left[\frac{2s+1}{s^2-3s+2}\right]$$

$$= 2L^{-1}\left[\frac{1}{s+1}\right] + L^{-1}\left[\frac{2s+1}{(s-3/2)^2-1/4}\right]$$

$$= 2e^{-t} + 2L^{-1}\left[\frac{s+1/2}{(s-3/2)^2-1/4}\right]$$

$$= 2e^{-t} + 2L^{-1}\left[\frac{s+1/2-2+2}{(s-3/2)^2-1/4}\right]$$

$$= 2e^{-t} + 2L^{-1}\left[\frac{s-3/2}{(s-3/2)^2-1/4}\right] + 4L^{-1}\left[\frac{1}{(s-3/2)^2-(1/2)^2}\right]$$

$$= 2e^{-t} + 2e^{3t/2}L^{-1}\left[\frac{s}{s^2-(1/2)^2}\right] + 4e^{3/2\,t}\frac{1}{1/2}\sinh 1/2$$

$$= 2e^{-t} + 2e^{3t/2}\cosh 1/2 + 8e^{3t/2}\sinh 1/2$$

Eg: 5 Find $L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right]$

Solu:-

$$\frac{1}{(s^2+a^2)(s^2+b^2)} = \frac{A}{s^2+a^2} + \frac{B}{s^2+b^2}$$

$$\frac{1}{(s^2+a^2)(s^2+b^2)} = \frac{A(s^2+b^2)+B(s^2+a^2)}{(s^2+a^2)(s^2+b^2)}$$

$$A(s^2+b^2)+B(s^2+a^2) = 1$$

putting $s^2 = -a^2$,

$$A((-a^2)+b^2) + B((-a^2)+a^2) = 1$$

$$A(-a^2+b^2) = 1$$

$$A(b^2-a^2) = 1$$

$$A = 1/b^2-a^2 = \frac{-1}{a^2-b^2}$$

Initial value Theorem:

If the laplace transforms of $f(t)$ & $f'(t)$ exists and $L[f(t)] = F(s)$, then $L[f(t)] = F(s)$, then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Proof: WKT,

$$L[f'(t)] = sL[f(t)] - f(0)$$

$$= sF(s) - f(0)$$

Taking limits on both sides as $s\to\infty$, we get,

$$\lim_{s\to\infty} L[f'(t)] = \lim_{s\to\infty} \{sF(s) - f(0)\}$$

i.e., $\lim_{s\to\infty} \{sF(s) - f(0)\} = \lim_{s\to\infty} L[f'(t)]$

$$= \lim_{s\to\infty} \int_0^\infty e^{-st} f'(t)\, dt$$

$$= 0 \quad \left[\because \lim_{s\to\infty} \int_0^\infty e^{-st} f(t)dt = 0\right]$$

$$\boxed{\therefore \lim_{s\to\infty} sF(s) = f(0) = \lim_{t\to 0} f(t)}$$

Solved problems using initial value Theorem:

eg. Verify the initial value theorem for the function $f(t) = ae^{-bt}$

Solu:-

Initial value theorem is

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

Here $f(t) = ae^{-bt}$ — (1)

$$\therefore F(s) = L[f(t)] = L[ae^{-bt}]$$

$$= a\frac{1}{s+b} \quad -(2)$$

$\therefore \lim_{t\to 0} f(t) = \lim_{t\to 0} a e^{-bt} = a$ (using ①) $\to$ (A)

$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s \cdot \frac{a}{s+b} \left(\frac{\infty}{\infty}\right)$ (using ②)

$= \lim_{s\to\infty} \frac{a}{1} = a \quad \to$ (B)

[using L'Hospital's rule]

From (A) & (B) we get,

$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Q2 Verify the initial value theorem for $f(t) = e^{-2t}\sin t$

Sol: Initial value theorem is

$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) \quad$ —(A)

Here $f(t) = e^{-2t}\sin t$

$\therefore \lim_{t\to 0} f(t) = \lim_{t\to 0} e^{-2t}\sin t$

$= 0 \quad$ —(1)

Now, $F(s) = L[f(t)]$

$= L[e^{-2t}\sin t]$

$= \left(\frac{1}{s^2+1}\right)_{s\to s+2}$

$= \frac{1}{(s+2)^2+1}$

$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{s}{(s+2)^2+1} \left(\frac{\infty}{\infty}\right)$

$= \lim_{s\to\infty} \frac{1}{2(s+2)}$ [using L'Hospital's rule]

$\frac{1}{\infty} = 0 \quad$ —(2)

From (1) & (2) we get,

$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Q1 Verify the initial value theorem for the function $f(t) = 1+e^{-t}+t^2$

Sol: Initial value Theorem is

$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Here $f(t) = 1+e^{-t}+t^2$

$\lim_{t\to 0} f(t) = 1+e^{0}+0 = 1+1$

$= 2 \quad$ —(1)

$\therefore F(s) = L[f(t)] = L[1+e^{-t}+t^2]$

$= \frac{1}{s} + \frac{1}{s+1} + \frac{2}{s^3}$

$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\left[\frac{1}{s} + \frac{1}{s+1} + \frac{2}{s^3}\right]$

$= \lim_{s\to\infty} \left[1 + \frac{s}{s+1} + \frac{2}{s^2}\right]$

$= \lim_{s\to\infty} 1 + \lim_{s\to\infty} \frac{s}{s+1} + \lim_{s\to\infty} \frac{2}{s^3}$

$= 1 + 1/1 + 0 = 2 \quad$ —(2)

[using L'Hospital's rule]

From (1) & (2) we get,

$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$.

Final Value Theorem

If the Laplace transforms of $f(t)$ & $f'(t)$ exist and $L[f(t)] = F(s)$, then $L[f(t)] = F(s)$, then $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$.

proof: WKT,

$L[f'(t)] = sL[f(t)] - f(0)$

$= sF(s) - f(0)$

i.e., $sF(s) - f(0) = \int_0^\infty e^{-st} f'(t)\,dt$

i.e., $\lim_{s\to 0} \{sF(s) - f(0)\} = \lim_{s\to 0} \int_0^\infty e^{-st} f'(t)\,dt$

$= \int_0^\infty f'(t)\,dt = [f(t)]_0^\infty$

$= \lim_{t\to\infty} f(t) - f(0)$

$\lim_{t\to 0} sF(s) - f(0) = \lim_{t\to\infty} f(t) - f(0)$

i.e., $\lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t)$

Solved problems using final value Theorem.

Ex:1 Verify final value Theorem for the function $f(t) = t^2 e^{-3t}$

solu: Final value theorem is

$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Here $f(t) = t^2 e^{-3t}$

$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} (t^2 e^{-3t})$

$= 0 \qquad (\because e^{-\infty} = 0) \quad -(1)$

$\therefore F(s) = L[f(t)] = L[t^2 e^{-3t}]$

$= \frac{d^2}{ds^2} L[e^{-3t}] = \frac{d^2}{ds^2}\left[\frac{1}{s+3}\right]$

$= \frac{d}{ds}\left[\frac{(s+3)0 - 1}{(s+3)^2}\right] = -\frac{d}{ds}\left[\frac{1}{(s+3)^2}\right]$

$= -\left[\frac{(s+3)^2 0 - 2(s+3)}{(s+3)^4}\right] = \frac{2}{(s+3)^3}$

$\lim_{t\to 0} sF(s) = \lim_{s\to 0} \frac{2s}{(s+3)^3}$

From (1) & (2), we get

$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Eg 2 Verify final value theorem for $f(t) = 1 - e^{-at}$

solu: Final value theorem is

$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Here $f(t) = 1 - e^{-at}$

$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} (1 - e^{-at})$

$= \lim_{t\to\infty} 1 - \lim_{t\to\infty} e^{-at}$

$= 1 - e^{-\infty} = 1 - 0 = 1 \quad -(1)$

$\therefore F(s) = L[f(t)] = L[1 - e^{-at}]$

$= \frac{1}{s} - \frac{1}{s+a} = \frac{s+a-s}{s(s+a)} = \frac{a}{s(s+a)}$

$\lim_{s\to 0} sF(s) = \lim_{s\to 0} s\frac{a}{s(s+a)}$

$= \lim_{s\to 0} \frac{a}{s+a} = \frac{a}{a} = 1 \quad -(2)$

From (1) & (2), we get

$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$.

Eg 3 Verify the initial & final value theorem for the function $f(t) = 1 + e^{-t}(\sin t + \cos t)$.

solu:

Final Value Theorem

$$\underset{t\to\infty}{Lt} f(t) = \underset{s\to 0}{Lt}\, sF(s)$$

L.H.S $\Rightarrow$ $= \underset{t\to\infty}{Lt} f(t)$

$= \underset{t\to\infty}{Lt} \{1 + e^{-t}(\sin t + \cos t)\}$ $(\because e^{-\infty}$

R.H.S $\Rightarrow$ $= \underset{s\to 0}{Lt}\, sF(s)$

$$= \underset{s\to 0}{Lt}\, s\left[\frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}\right]$$

$$= \underset{s\to 0}{Lt}\left[1 + \frac{s}{(s+1)^2+1} + \frac{s(s+1)}{(s+1)^2+1}\right]$$

$$= 1 + \frac{0}{(0+1)^2+1} + \frac{0(0+1)}{(0+1)^2+1}$$

$$= 1 + \frac{0}{2} + \frac{0}{2}$$

$$= 1$$

L.H.S = R.H.S

Hence final value theorem is verified.

Applications of Laplace transforms for solving Differential Equations.

Ex: 1 Solve $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 5y = 5$ given that $y=0$, $\frac{dy}{dt} = 2$, when $t=0$.

Solu:

Given differential equation can be written as.

$$y''(t) + 4y'(t) - 5y(t) = 5$$

Taking Laplace transform on both sides, we get

$L[y''(t)] + 4L[y'(t)] - 5L[y(t)] = L[5]$

$s^2 L[y(t)] - s y(0) - y'(0) + 4[s L[y(t)] - y(0)] - 5 L[y(t)] = L[5]$ ... (1)

given $y(0) = 0$, $y'(0) = 2$ ... (2)

Let us denote $L[y(t)] = \bar{y}$ (for convenience) ... (3)

Substituting (2) and (3) in (1), we get

$s^2 L[y(t)] + 4 s L[y(t)] - 5 L[y(t)] = 5/s$

$s^2 \bar{y} + 4s\bar{y} - 5\bar{y} = 5/s$

$s^2 \bar{y} + 4s\bar{y} - 5\bar{y} = 5/s + 2$

i.e. $(s^2 + 4s - 5)\bar{y} = 5/s + 2$

$$\bar{y} = \frac{5/s + 2}{(s^2+4s-5)}$$

$$= \frac{\frac{5+2s}{s}}{(s^2+4s-5)} = \frac{5+2s}{s(s^2+4s-5)} \quad (4)$$

Using partial fraction,

$$\frac{2s+5}{s(s^2+4s-5)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s-5} \quad (5)$$

$$\frac{2s+5}{s(s^2+4s-5)} = \frac{A(s^2+4s-5) + (Bs+C)s}{s(s^2+4s-5)}$$

$2s+5 = A(s^2+4s-5) + (Bs+C)s$

Putting $s = 0$,

$2(0)+5 = A(0+4(0)-5) + (B(0)+C)(0)$

$5 = A(0-5) + 0$

$5 = -5A$

$\boxed{A = -1}$

Equating coefficient of s^2,

$A + B = 0$

$-1 + B = 0$

$\boxed{B = 1}$

Equating coefficient of s,

$4A + C = 2$

$4(-1) + C = 2$

$-4 + C = 2$

$C = 2 + 4 \Rightarrow \boxed{C = 6}$

Substitute the values of A, B, C in eqn (5), we get

$$\frac{2s+5}{s(s^2+4s-5)} = \frac{-1}{s} + \frac{(1)s+6}{s^2+4s-5}$$

$$\bar{y} = \frac{-1}{s} + \frac{s+6}{s^2+4s-5}$$

i.e. $L[y(t)] = \frac{-1}{s} + \frac{s+6}{s^2+4s-5}$ (using (4))

$$y(t) = L^{-1}\left[\frac{-1}{s} + \frac{s+6}{s^2+4s-5}\right]$$

$$= L^{-1}\left[\frac{-1}{s}\right] + L^{-1}\left[\frac{s+6}{s^2+4s-5}\right]$$

$$= -1 + L^{-1}\left[\frac{s+2+4}{(s+2)^2-9}\right]$$

$$= -1 + L^{-1}\left[\frac{s+2}{(s+2)^2-3^2}\right] + 4 L^{-1}\left[\frac{1}{(s+2)^2-3^2}\right]$$

$$= -1 + e^{-2t} L^{-1}\left[\frac{s}{s^2-3^2}\right] + 4e^{-2t} L^{-1}\left[\frac{1}{s^2-3^2}\right]$$

$$= -1 + e^{-2t}\cosh 3t + 4e^{-2t}\frac{\sinh 3t}{3}$$

Q. Solve $y'' - 3y' + 2y = e^{2t}$, $y(0) = -3$, $y'(0) = 5$

Sol: Given $y'' - 3y' + 2y = e^{2t}$

Taking Laplace transform both sides

$$L[y''] - 3L[y'] + 2L[y] = L[e^{2t}]$$

$$[s^2 L[y(t)] - s y(0) - y'(0)] - 3[sL[y(t)] - y(0)] + 2L[y(t)] = L[e^{2t}] \rightarrow (A)$$

Substituting $y(0) = -3$, $y'(0) = 5$ in (A)

& putting $L[y(t)] = \bar{y}$, we get

$$s^2\bar{y} + 3s - 5 - 3[s\bar{y} + 3] + 2\bar{y} = \frac{1}{s-2}$$

$$s^2\bar{y} + 3s - 5 - 3s\bar{y} - 9 + 2\bar{y} = \frac{1}{s-2}$$

i.e. $$s^2\bar{y} + 3s - 3s\bar{y} + 2\bar{y} - 14 = \frac{1}{s-2}$$

$$\bar{y}(s^2 - 3s + 2) + 3s - 14 = \frac{1}{s-2}$$

$$\bar{y}(s^2 - 3s + 2) = \frac{1}{s-2} - 3s + 14$$

$$\bar{y}(s^2 - 3s + 2) = \frac{1 - 3s(s-2) + 14(s-2)}{s-2}$$

$$\bar{y}(s^2 - 3s + 2) = \frac{1 - 3s^2 + 6s + 14s - 28}{s-2}$$

$$\bar{y}(s^2 - 3s + 2) = \frac{-3s^2 + 20s - 27}{s-2}$$

$$\therefore \bar{y} = \frac{-3s^2 + 20s - 27}{(s-2)(s^2 - 3s + 2)}$$

i.e. $$L[y(t)] = \frac{-3s^2 + 20s - 27}{(s-2)(s-1)(s-2)}$$

$$L[y(t)] = \frac{-3s^2 + 20s - 27}{(s-1)(s-2)^2}$$

$$y(t) = L^{-1}\left[\frac{-3s^2 + 20s - 27}{(s-1)(s-2)^2}\right]$$

Taking partial fraction, now

$$\frac{-3s^2 + 20s - 27}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$\frac{-3s^2 + 20s - 27}{(s-1)(s-2)^2} = \frac{A(s-2)^2 + B(s-1)(s-2) + C(s-1)}{(s-1)(s-2)^2} \rightarrow (A)$$

$$-3s^2 + 20s - 27 = A(s-2)^2 + B(s-1)(s-2) + C(s-1)$$

putting $s = 1$,

$$-3(1)^2 + 20(1) - 27 = A(1-2)^2 + B(1-1)(1-2) + C(1-1)$$

$$-3 + 20 - 27 = A(-1)^2$$

$$A = -30 + 20 \Rightarrow \boxed{A = -10}$$

putting $s = 2$,

$$-3(2)^2 + 20(2) - 27 = A(2-2)^2 + B(2-1)(2-2) + C(2-1)$$

$$-12 + 40 - 27 = C$$

$$\boxed{C = 1}$$

Equating coefficient of s^2,

$$A + B = -3$$

$$-10 + B = -3$$

$$B = -3 + 10 \Rightarrow \boxed{B = 7}$$

Substituting in (A) ⇒

$$\frac{-3s^2 + 20s - 27}{(s-1)(s-2)^2} = \frac{-10}{s-1} + \frac{7}{s-2} + \frac{1}{(s-2)^2}$$

$$L^{-1}\left[\frac{-3s^2 + 20s - 27}{(s-1)(s-2)^2}\right] = -10L^{-1}\left[\frac{1}{s-1}\right] + 7L^{-1}\left[\frac{1}{s-2}\right] + L^{-1}\left[\frac{1}{(s-2)^2}\right]$$

$$= -10e^{t} + 7e^{2t} + L^{-1}\left[\frac{1}{(s-2)^2}\right]$$

$= -10e^{t} + 7e^{2t} + e^{2t} L^{-1}\left[\frac{-1}{s^2}\right]$

$= -10e^{t} + 7e^{2t} + t \cdot e^{2t}$

Ex: 3 Solve $y'' + 2y' - 3y = \sin t$ given, $y=0$, $y'(0)$ when $t=0$

Solu:

Given, $y''(t) + 2y'(t) - 3y(t) = \sin t$.

Taking laplace transform on bothside, we get,

$L[y''(t)] + L[2y'(t)] - L[3y(t)] = L[\sin t]$

$s^2 L[y(t)] - s[y(0)] - y'(0) + 2\{s L[y(t)] - y(0)\}$

$-3 L[y(t)] = \frac{1}{s^2+1}$

putting $y(0)=0$, $y'(0)=0$ & $L[y(t)] = \bar{y}$,

we get

$s^2 \bar{y} + 2s\bar{y} - 3\bar{y} = \frac{1}{s^2+1}$

$\bar{y}(s^2 + 2s - 3) = \frac{1}{s^2+1}$

$\bar{y} = \frac{1}{(s^2+2s-3)(s^2+1)}$

$\bar{y} = \frac{1}{(s-1)(s+3)(s^2+1)}$

$L[y(t)] = \frac{1}{(s-1)(s+3)(s^2+1)}$

i.e., $[y(t)] = L^{-1}\left[\frac{1}{(s-1)(s+3)(s^2+1)}\right] \to (1)$

Taking partial fraction we get,

$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1} \to (A)$

$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A(s+3)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+3)}{(s-1)(s+3)(s^2+1)}$

$1 = A(s+3)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s+3)(s-1)$

putting $s=1$,

$1 = A(1+3)(1^2+1) + B(1-1)(1^2+1) + (C(1)+D)(1+3)(1-1)$

$1 = A(4)(2) \Rightarrow 8A = 1$

$\therefore \boxed{A = 1/8}$

putting $s=-3$,

$1 = A(-3+3)((-3)^2+1) + B((-3)-1)((-3)^2+1) + (C(-3)+D)(-3+3)(-3-1)$

$1 = B(-4)(10) \Rightarrow -40B = 1$

$\therefore \boxed{B = -1/40}$

Equating coefficient of s^3,

$A + B + C = 0$

$C = -A - B$

$= -1/8 + 1/40 = \frac{-5+1}{40} = \frac{-4}{40}$

$\boxed{C = -1/10}$

Equating constant coefficients,

$3A - B - 3D = 1$

$3(1/8) - (-1/40) - 3D = 1$

$-3D = 1 - 3/8 - 1/40$

$3D = 3/8 + 1/40 - 1$

$3D = \frac{15+1-40}{40} = \frac{-24}{40}$

$3D = -24/40 \Rightarrow D = \frac{-24}{3 \times 40}$

$$\boxed{D = -1/5}$$

(A) $\Rightarrow$ substituting we get,

$$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{1/8}{s-1} + \frac{-1/40}{s+3} + \frac{-1/10(s)}{s^2+}$$

$$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{1/8}{s-1} + \frac{-1/40}{s+3} + \frac{(-1/10)(s+2)}{s^2+1}$$

$$\therefore L^{-1}\left[\frac{1}{(s-1)(s+3)(s^2+1)}\right]$$

$$= 1/8\, L^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{40} L^{-1}\left[\frac{1}{s+3}\right] - \frac{1}{10} L^{-1}\left[\frac{s+}{s^2+}\right]$$

$$= 1/8\, e^{-t} - 1/40\, e^{-3t} - \frac{1}{10}\left\{L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{2}{s^2+}\right]\right.$$

$$= 1/8\, e^{-t} - 1/40\, e^{-3t} - \frac{1}{10}\left[\cos t + 2\sin t\right]$$