closed sets and limit points.

Closed sets :-

A subset A of a topological space x is said to closed if the set x-A is open.

Ex In the plane R^2, the set $\{x \times y \mid x \geq 0 \text{ and } y \geq 0\}$ is closed because its complement is the union of the two sets. $(-\infty, 0) \times R$ and $R \times (-\infty, 0)$. each of which is a product of open sets of R and is, therefore, open in R^2.

Thm 17.1

Im (2)

Let x be a topological space. then the following conditions hold:- (i) ϕ and x are closed

(ii) Arbitrary intersection of closed set are closed.

(iii) Finite unions of closed sets are closed.

proof:

i) ϕ and x are closed because they are the complements of the open sets x and ϕ, respectively.

ii) Given a collection of closed sets $\{A_\alpha\}_{\alpha \in J}$.

Apply De Morgan's law.

$$X - \bigcap_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} (X - A_\alpha)$$

$\because$ the sets $X - A_\alpha$ are open by defn.

The right side of this equation represents an arbitrary union of open sets, and is thus open.

$\therefore \bigcap A_\alpha$ is closed.

③ III^{ly}

if A_i is closed for $i=1,2,\dots n$ consider the equation

$$X-\bigcup_{i=1}^{n} A_i=\bigcap_{i=1}^{n}(X-A_i)$$

The set on the right side of this eqn is a finite intersection of open sets and is therefore open.

Hence $\cup A_i$ is closed.

Thm 17.2

Let Y be a subspace of X. Then a set A is closed in Y iff it equals the intersection of a closed set of X with Y.

proof:

Assume that $A=C\cap Y$, where C is closed in X.

Then $X-C$ is open in X, so that $(X-C)\cap Y$ is open in Y, by defn of the subspace topology.

But $(X-C)\cap Y=Y-A$.

Hence $Y-A$ is open in Y, so that A is closed in Y.

Conversely,

Assume that A is closed in Y.

Then $Y-A$ is open in Y. So that by defn it equals the intersection of an open set U of X with Y.

The set $X-U$ is closed in X and $A=Y\cap(X-U)$, So that A equals the intersection of a closed set of X with Y as desired.

closure and interior of a set.

Given a subset A of a topological space X, the interior of A is defined as the union of all open sets contained in A. and

The closure of A is defined as the intersection of all closed sets containing A.

The interior of A is denoted by $\text{Int } A$ and the closure of A is denoted by $Cl\, A$ (or) by $\bar{A}$.

obviously $\text{Int } A$ is an open set and $\bar{A}$ is a closed set.

$\text{Int } A \subset A \subset \bar{A}$.

If A is open $A = \text{Int } A$; while if A is closed $A = \bar{A}$.

Thm 17.4:

Let Y be a subspace of X; let A be a subset of Y; Let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$.

proof:

Let B denote the closure of A in Y.

The set $\bar{A}$ is closed in X, so $\bar{A} \cap Y$ is closed in Y by (Thm 17.2)

$\therefore$ $\bar{A} \cap Y$ contains A, and since by defn. B equals the intersection of all subsets of Y containing A,

we must have $B \subset (\bar{A} \cap Y)$.

on other hand, we know that B is closed in Y.

Hence by thm 17.2,

$B = C \cap Y$ for some set C closed in X. Then C is a closed set X containing A;

because $\bar{A}$ is the intersection of all such closed sets.

we conclude that $\bar{A} \subset C$. Then $(\bar{A} \cap Y) \subset (C \cap Y) = B$.

Thm 17.5

Let A be a subset of the topological space X.

(a) Then $x \in \bar{A}$ if and only if every open set U containing x intersects A.

(b) Supposing the topology of X is given by a basis, then $x \in \bar{A}$ iff every basis element B containing x intersects A.

proof:

Consider the Statement in (a).

it is a Statement of the form $P \Leftrightarrow Q$.

Let us transform each implication to its contrapositive, Thereby obtaining the logically equivalent statement

$(\text{not } P) \Leftrightarrow (\text{not } Q)$.

it is the following.

$x \notin \bar{A} \Leftrightarrow$ there exists an open set U containing x that does not intersect A.

If x is not $\bar{A}$, the set $U = X - \bar{A}$ is an open set containing x that does not intersect A, as desired.

Conversely,

if there exists an open set U containing x which does not intersect A, then $X - U$ is a closed set containing A.

By defn of the closure $\bar{A}$, The set $X - U$ must contain $\bar{A}$; Therefore x cannot be in $\bar{A}$.

Statement (b) follows readily,

If every open set containing x intersects A, so does every basis element B containing x, because B is an open set.

Conversely,

if every basis element containing x intersects A, so does every open set U containing x, because U contains a basis element that contains x.

Limit points :-

If A is a subset of the topological space X and if x is a point of X, we say that x is a **limit point** (or "**cluster point**" or "**point of accumulation**") of A if every neighborhood of x intersects A in some point other than x itself.

Said differently, x is a limit point of A if it belongs to the closure of $A - \{x\}$. The point x may lie in A or not.

Thm 17.6

Let A be a subset of the topological space X; let A' be the set of all limit points of A. Then $\bar{A} = A \cup A'$.

proof:-

If x is in A', every neighborhood of x intersects A.

$\therefore$ by thm (17.5), $x \in \bar{A}$. Hence $A' \subset \bar{A}$.

$\because$ by defn) $A \subset \bar{A}$ it follows that $A \cup A' \subset \bar{A}$.

To demonstrate the reverse inclusion, we let x be a point of $\bar{A}$ and show that $x \in A \cup A'$.

If x happens to lie in A, it is trivial that $x \in A \cup A'$; suppose that x does not lie in A.

Since $x \in \bar{A}$, W.K.T every neighborhood U of x intersects A; because $x \notin A$, the set U must intersect A

in a point different from x. Then $x \in A'$. So that $x \in A \cup A'$.

Corollary 17.7:

A subset of a topological space is closed iff it contains all its limit points.

proof:

The set A is closed if and only if $A = \bar{A}$, and the latter holds iff $A' \subset A$.

Hausdorff spaces:-

A topological space X is called a Hausdorff space if for each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1 and U_2 of x_1 and x_2, respectively, that are disjoint.

Thm 17.8 Im.

Every finite point set in Hausdorff space X is closed.

proof:

It suffices to S.T every one-point set $\{x_0\}$ is closed. If x is a point of X different from x_0, then x and x_0 have disjoint neighborhoods U and V, respectively.

Since U does not intersect $\{x_0\}$, the point x cannot belong to the closure of the set $\{x_0\}$.

As a result, the closure of the set $\{x_0\}$ is $\{x_0\}$ itself, so that is closed.

Thm 17.9

Let X be a space satisfying the T_1 axiom; let A be a subset X. Then the point x is a limit point of A iff every neighbourhood of x contains infinitely many points of A.

Proof:

If every neighbourhood of x intersects A in infinitely many points, it certainly intersects A in some point other than x itself, so that x is a limit point of A.

Conversely,

Suppose that x is a limit point of A, and suppose some neighbourhood U of x intersects A in only finitely many points.

Then U also intersects $A-\{x\}$ in finitely many points, let $\{x_1, \dots x_n\}$ be the points of $U \cap (A-\{x\}))$.

The set $X-\{x_1, \dots . x_m\}$ is an open set of X.

Since the finite point set $\{x_1, \dots x_m\}$ is closed then

$$U \cap (X-\{x_1 \dots x_m\}),$$

is a neighbourhood of x that intersects the $A-\{x\}$ not at all.

This contradicts the assumption that x is a limit point of A.

Thm: 17.10

Continuity of a function:-

Let X and Y be topological spaces. A function $f: X \to Y$ is said to be continuous if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

Recall that $f^{-1}(V)$ is the set of all points x of X for which $f(x) \in V$; it is empty if V does not intersect the image set $f(X)$ of f.

Thm 18.1

Let X and Y be topological spaces; let $f: X \to Y$. Then the following are equivalent.

1) f is continuous

2) For every subset A of X, one has $f(\bar{A}) \subset \overline{f(A)}$.

3). For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.

4). For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.

If the condition in (4) holds for the point x of X, we say that f is continuous at the point x.

proof:

We show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) and that (1) $\Rightarrow$ (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2).

Assume that f is continuous. Let A be a subset of X. We show that if $x \in \bar{A}$, then $f(x) \in \overline{f(A)}$.

Let V be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is an open set of X containing x; it must intersect A in some point y.

Then V intersects $f(A)$ in the point $f(y)$,

so that $f(x) \in \overline{f(A)}$,

② ⇒ ③ :

Let B be closed in Y and let $A = f^{-1}(B)$.

TO P.T A is closed in X,

we S.T $\bar{A} = \bar{A}$. By elementary set theory,

we have $f(A) = f(f^{-1}(B)) \subset B$,

$\therefore$ if $x \in \bar{A}$.

$$f(x) \in f(\bar{A}) \subset \overline{f(A)} \subset \bar{B} = B,$$

So that $x \in f^{-1}(B) = A$.

Thus $\bar{A} \subset A$. So that $\bar{A} = A$.

③ ⇒ ①.

Let V be an open set of Y. Set $B = Y - V$.

Then $f^{-1}(B) = f^{-1}(Y) - f^{-1}(V)$

$= X - f^{-1}(V)$.

Now, B is a closed set of Y. Then $f^{-1}(B)$ is closed in X by hypothesis, So that $f^{-1}(V)$ is open in X,

① ⇒ ④.

Let $x \in X$ and let V be a neighbourhood of $f(x)$.

Then the set $U = f^{-1}(V)$ is neighbourhood of x such that $f(U) \subset V$.

④ ⇒ ①

Let V be an open set of Y. Let x be a point of $f^{-1}(V)$.

Then $f(x) \in V$, So that by hypothesis there is neighbourhood U_x of x such that $f(U_x) \subset V$. Then $U_x \subset f^{-1}(V)$.

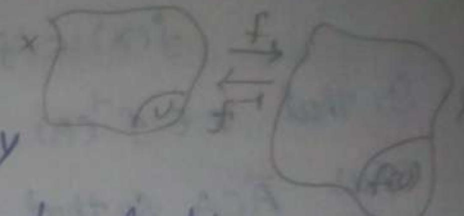
It follows that $f^{-1}(V)$ can be written as the union of the open sets U_x, so that it is open.

Homeomorphisms:

Let x and y be topological spaces; let $f: x \to y$ be a bijection. If both the function f and the inverse function $f^{-1}: y \to x$ are continuous, then f is called a homeomorphism.

<u>Remark</u>:

S.T a homeomorphism $f: x \to y$



gives us a bijective correspondence not only b/w x and y but between the collections of open set of x and of y.

<u>Result</u>:

Any a result, any property of x that is entirely expressed in terms of the topology of x yields, via the correspondence f, the corresponding property for the space y. Such a property of x is called a topological property of x.

<u>Defn</u>:-

Now $f: x \to y$ is an injective continuous map, where x and y are topological space.

Let z be the image set $f(x)$.

Considered as a subspace of y; then the fun:- $f': x \to z$ obtained by restricting the range of f is bijective.

If f' happens to be a homeomorphism of x with z, we say that the map $f: x \to y$ is a topological imbedding or simply an imbedding of x in y.

Thm 18.2

Rules for constructing continuous functions

Let X, Y, and Z be topological spaces.

(a) (Constant fun.) If $f : X \to Y$ maps all of X into the single point y_0 of Y, then f is continuous.

(b) (Inclusion) If A is a subspace of X, the inclusion fun. $j : A \to X$ is continuous.

(c) (Composites) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f : X \to Z$ is continuous.

(d) (Restricting the domain) If $f : X \to Y$ is continuous, and if a subspace of X, then the restricted function $f/A : A \to Y$ is continues.

(e) (Restricting or expanding the range) Let $f : X \to Y$ be continues. If Z is a subspace of Y containing the image set $f(X)$. then the function $g : X \to Z$ obtained by restricting the range of is continuoes. If Z is a space having Y as a susface, the the function $h : X \to$ Z is a suspace, then the function $h : X \to Z$ obtained by expaniting the range of f is continues.

(f) (Local formulation of continuity) the map $f : X \to Y$ is continuios if X can be written as union of open sets U_α such that f/U_α is continuoes for each α.

Proof: (a) Let $f(x) = y_0$ for every x in X. Let V be open in Y. The set $f^{-1}(V)$ equal X or $\emptyset$ depending on whether V contians y_0 or not. In either case. it is open. ~~the suspace topology.~~

(b) If U is open in X then $j^{-1} = U \cap A$, which is open A by definfion of the suspace topology.

(c) ~~If V is open in X, then $f^{-1}(V) = V \cap A$. which is open in A by defintion~~ of

(c) If U is open in Z then $g^{-1}(U)$ is open Y in and $f^{-1}(g^{-1}(U))$ is open in X ~~open~~ But.

$$f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U).$$

by elementry set theory.

(d) The function f/A equals the composite of the inclusion map $j: A \to X$ and the map $f: X - Y$. ~~be~~ bothe of which are continuous.

(e) Let $f: X \to Y$ be continouy. If $f(X) \subset Z \supset Y$, we show that the function $g: X \to Z$ obtained from f is continoues. Let Let Be open in in Z. Then $B = Z \cap U$ for Some open Set U of Y. Because Z contains the entire image set $f(X)$.

$$f^{-1}(U) = g^{-1}(B)$$

by elamentry set theory: since $f^{-1}(U)$ is open. so is g^{-1} B.

To show $h: X \to Z$ is continouy if Z has Y as a suspace. note that h is the composite of map $f : X \to Y$ and the inclusion map $j : Y \to Z$.

(f) By hypothesis, we can write X as a union of open sets U_α, such that $f|U_\alpha$ is continous for each α. Let $f : X \to$ be continous Let B be open in Z. Then $B = Z \cap U$ for some open set U of Y. Because Z contians the entire image set $f(X)$

$$f^{-1}(U) = g^{-1}(B)$$

because both expressive represent the sets of of those points x lying in U_α for which $f(x) \in V$. Since $f|U_\alpha$ is continoues, this set is open in U_α. and hence open in X. But $f(x) \in V$. Since f/U_α is and hense open is in X. But $f(x)$

$$f^{-1}(V) \cap U_\alpha = (V) \cap U_\alpha)$$

So that $f^{-1}(V)$ is also open in X.

Thm 18.3 (The pasting lemma)

Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \to Y$, defined by setting $h(x) = f(x)$ if $x \in A$, and $h(x) = g(x)$ if $x \in B$.

proof

Let C be a closed subset of Y.

Now

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$$

by elementary set theory.

Since f is continuous, $f^{-1}(C)$ is closed in A, and Therefore, closed in X.

III^ly

$g^{-1}(C)$ is closed in B and $\therefore$ closed in X.

Their union $h^{-1}(C)$ is thus closed in X.

Thm 18.4 (Maps into products)

Let $f: A \rightarrow X \times Y$ be given by the equation

$$f(a) = (f_1(a), f_2(a)).$$

Then f is continuous iff functions $f_1 : A \rightarrow X$ and $f_2 : A \rightarrow Y$. are Continuous.

The maps f_1 and f_2 are called the Coordinate functions of f.

Proof:-

Let $\pi_1 : X \times Y \rightarrow X$ and $\pi_2 : X \times Y \rightarrow Y$ be projections onto the 1st and 2nd factors, respectively,

These maps are continuous.

For $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$, and these sets are open if U and V are open. for each $a \in A$.

$$f_1(a) = \pi_1(f(a)) \text{ and } f_2(a) = \pi_2(f(a)).$$

If the funl f is continuous, then f_1 and f_2 are composites of continuous funl and therefore Continuous.

Conversely,

Suppose that f_1 and f_2 are Continuous.

We S.T for each basis element $U \times V$ for the topology of $X \times Y$, its inverse image $f^{-1}(U \times V)$ is open.

A point a is in $f^{-1}(U\times V)$ if and only if $f(a)\in U\times V$ that is iff $f_1(a)\in U$ and $f_2(a)\in V$.

$\therefore f^{-1}(U\times V) = f_1^{-1}(U)\cap f_2^{-1}(V)$.

Since both of the sets $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open, so is their intersection.

The product Topology.

P.8 113 ↓ 12.3. Defn:-

Let J be an index set. Given a set X, we define a J-tuple of elements of X to be a function $x : J\to X$. If α is an element of J, we often denote the value of x at α by x_α rather than $x(\alpha)$ we call it the αth Coordinate of x, and we often denote the function the x itself by the symbol.

$$(x_\alpha)_{\alpha\in J},$$

Defn:- which is as close as we can come to a "tuple notation" for an arbitrary index set J. we denote the set of all J-tuples of element of X by X^J.

Defn:- Let $(A_\alpha)_{\alpha\in J}$ an indexed family of area sets ; Let $X = \bigcup_{\alpha\in J} A_\alpha$ for each $\alpha\in J$. That is it is the set of all functions.

$$x : J\to \bigcup_{\alpha\in J} A_\alpha$$

Such that $x(\alpha) \in A_\alpha$ for each $\alpha \in J$

Defn:- Let $(X_\alpha)_{\alpha \in J}$ be an indexed family of topological spaces. Let us take us a basis for a topology on the product form

$$\prod_{\alpha \in J} X_\alpha$$

the collection of all sets of the form.

$$\prod_{\alpha \in J} U_\alpha,$$

where U_α is open in X_α, for each $\alpha \in J$. The topology generated by this basis is called the box topology.

Defn:- Let S_β denote the collection

$$S_\beta = (\pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta)$$

and Let S denote the union of these collections

$$S = \bigcup_{\beta \in J} S_\beta$$

The topology generated by the subbasis S is called Product topology. In this topology $\prod_{\alpha \in J} X_\alpha$ is called a product space.

Theorem 19.1 (comparison of the box and product topologies) the box topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α. The product topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$ where, U_α is open in X_α for each α and U_α equals X_α. expect for finitely many values of α.

Theorem: 19.2.

Suppose The topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form.

$$\prod_{\alpha \in J} B_\alpha$$

Where $B_\alpha \in \mathcal{B}_\alpha$. for each α, will serve as a basis for the box topology on $\prod_{\alpha \in J} X_\alpha$.

the collection of all sets of the same form where, $B_\alpha \in \mathcal{B}_\alpha$ for finitely many indices α and $B_\alpha = X_\alpha$ for all the remaining indices. will serve as a basis for the product topology $\prod_{\alpha \in J} X_\alpha$.

Thm 19.6

Let $f: A \to \prod_{\alpha \in J} X_\alpha$ be given by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J},$$

where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then the fun!- continuous iff each fun!- f_α is continuous.

Proof

Let π_β be the projection of the product onto its β^{th} factor. The function π_β is continuous, for if U_β is open in X_β. The set $\pi_\beta^{-1}(U_\beta)$ is a subbasis element for the product topology on X_α.

Now suppose that $f: A \to \prod X_\alpha$ is continuous. The fun!- f_β equals the composite $\pi_\beta \circ f$; being the composite of two continuous fun!-, it is continuous.

Conversely,

Suppose that each co-ordinate fun!- f_α is continuous. T.P.T f is continuous, it suffices to P.T the inverse image under f of each subbasis element is open in A; we defined continuous fun!-.

A typical subbasis element for the product topology on $\prod X_\alpha$ is a set of the form $\pi_\beta^{-1}(U_\beta)$, where β is some index and U_β is open in X_β.

Now,

$$f^{-1}(\pi_\beta^{-1}(U_\beta)) = f_\beta^{-1}(U_\beta).$$

because $f_\beta = \pi_\beta \circ f$. Since f_β is continuous, This set is open in A.