UNIT - III

Connectedness:

Connected spaces:-

Defn:-

Let X be a topological space. A separation of X is pair U, V of disjoint nonempty open subsets of X whose union in X. The space X is said to be connected if there does not exist a separation of X.

Connectedness:-

The Property of the space $[a, b]$ on which the intermediate value depends is the property called Connectedness.

Lemma 23.1:

If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union Y, neither of which contains a limit point of the other. the space Y is connected if there exists no separation of Y.

proof:-

Suppose for 1^{st} that A and B form a separation of Y. Then A is both open and closed in Y.

The closure of A in Y is the set $\bar{A} \cap Y$ (where $\bar{A}$ as usual denotes the closure of A in X).

Since A is closed in Y, $A = \bar{A} \cap Y$; (or) $\bar{A} \cap B = \phi$.

$\therefore \bar{A}$ is the union of A and its limit points,

B contains no limit point of A.

A Similar argument S.T A contains no limit points of B.

Conversely,

Suppose that A and B are disjoint nonempty Sets whose union is Y, neither of which Contains a limit Point of the other. Then $\bar{A} \cap B = \phi$ and $A \cap \bar{B} = \phi$.

$\therefore$ we conclude that $\bar{A} \cap Y = A$ and $\bar{B} \cap Y = B$.

Thus both A and B are closed in Y. and

$\therefore$ $A = Y - B$ and $B = Y - A$, $\because$

They are open in Y as well.

Lemma 23.2

If the sets C and D form a separation of X, and if Y is a connected subspace of X, then Y lies entirely within either C or D.

Proof:

Since C and D are both open in X, the sets $C \cap Y$ and $D \cap Y$ are open in Y.

These two sets are disjoint and their union is Y, if they were both nonempty.

They would Constitute a separation of Y.

$\therefore$ one of them is empty,

Hence Y must lie entirely in C or in D.

Thm 23.3

The union of a collection of connected subspaces of X that have a point in common is connected.

proof:-

Let $\{A_\alpha\}$ be a collection of connected subspace of a space X;

let p be a point of $\cap A_\alpha$.

W.P.T The space $Y = \cup A_\alpha$ is connected.

Suppose that $Y = C \cup D$. is a separation of Y.

The point p is in one of sets C or D:

Suppose $p \in C$. since A_α is connected, it must lie entirely in either C or D. and it cannot lie in D. because it contains the point p of C.

Hence $A_\alpha \subset C$ for every α, so that $\cup A_\alpha \subset C$, contradicting the fact that D is nonempty.

Thm 23.4:

Let A be a connected subspace of X. if $A \subset B \subset \bar{A}$, Then B is also connected: (or)

If B is formed by adjoining to the connected subspace A some or all of its limit points, then B is connected

proof:

Let A be connected and let $A \subset B \subset \bar{A}$.

Suppose that $B = C \cup D$ is a separation of B.

By lemma: 23.2.

" the set A must lie entirely in C or in D.

Suppose that $A \subset C$. Then $\bar{A} \subset \bar{C}$;

Since $\bar{C}$ and D are disjoint, B cannot intersect D.

This contradicts the fact that D is a nonempty subset of B.

Thm: 23.5

The image of a connected space under a continuous map is ~~non~~. ~~no~~ connected.

Proof:

Let $f: X \to Y$ be a continuous map. Let X be connected.

To P.T image space $Z = f(X)$ is connected.

Since the map obtained from f by restricting its range to the space Z is also continuous ∴

it suffices to consider the case of a continuous surjective map.

$$g: X \to Z.$$

Suppose that $Z = A \cup B$ is a separation of Z into two disjoint nonempty sets open in Z.

Then $g^{-1}(A)$ and $g^{-1}(B)$ are disjoint set. whose union is X,

They are open in X because g is continuous and nonempty because g is surjective

∴ They form a separation of X. Contradicting the assumption that X is connected.

Thm 23.6

A finite Cartesian product of connected spaces is connected.

Proof:

We prove the thm first for the product of two connected spaces X and Y.

Choose a "base point" $a \times b$ in the product $X \times Y$.

Note that the "horizontal slice" $X \times b$ is connected, being homeomorphic with X, and each "vertical slice" $x \times Y$ is connected, being homeomorphic with Y.

As the result each "T-shaped" space

$$T_x = (X \times b) \cup (x \times Y).$$

is connected, being the union of two connected spaces that have the point $x \times b$ in common.

(23.2) The set A must lie entirely in C or in D"

Now form the union $\bigcup_{x \in X} T_x$ of all those T-shaped spaces.

This union is connected because it is the union of collection of connected spaces that have the point $a \times b$ in common.

Since this union equals $X \times Y$, the space $X \times Y$ is connected.

The Proof for any finite product of connected spaces follows by induction.

using the fact that $X_1 \times \dots \times X_n$ is homeomorphic with $(X_1 \times \dots \times X_{n-1}) \times X_n$.

24. Connected subspaces of the Real line.

Definition:

Linear continuum:

A simply ordered set L having more than one element is called a linear Continuum if the following hold.

i) L has the least upper bound property.

ii) if $x < y$, there exists z such that $x < z < y$.

Thm: 24.1:

If L is a linear continuum in the order topology, then L is Connected and so are intervals and rays in L.

Proof:

Recall that a subspace Y of L is said to be convex if for every pair of points a, b of Y with $a < b$, the entire interval $[a, b]$ of points of L lies in Y.

We prove that:

If Y is a Convex subspace of L, then Y is Connected.

so suppose that Y is the union of the disjoint nonempty sets A and B, each of which is open in Y.

choose $a \in A$ and $b \in B$.

suppose for convenience that $a < b$.

The interval $[a, b]$ of points of L is contained in Y.

Hence $[a, b]$ is the union of the disjoint sets.

$$A_0 = A \cap [a, b] \text{ and } B_0 = B \cap [a, b],$$

each of which is open in $[a, b]$ in the subspace topology.

which is the same order topology.

The sets A_0 and B_0 are nonempty because $a \in A_0$ and $b \in B_0$.

Thus, A_0 and B_0 constitute a separation of $[a, b]$.

Let $c = \sup A_0$.

$c \in A_0$ (or) B_0

we S.T c belongs to neither A_0 nor to B_0.

Which contradicts that fact that $[a, b]$ is the union of A_0 and B_0.

Case. 1.

suppose that $c \in B_0$. Then $c \neq a$, so either $c = b$ (or) $a < c < b$.

In either case,

It follows from the fact that B_0 is open in $[a, b]$ that there is some interval of the form $(d, c]$ contained in B_0. If $c = b$,

we have a contradiction at once, for d is a smaller upper bound on A_0 than c.

If $c < b$, we note that $(c, b]$ does not intersect A_0 (because c is an upper bound A_0). Then

$$(d, b] = (d, c] \cup (c, b]$$

does not intersect A_0.

Again, d is smaller upper bound on A_0 than c, contrary to construction.

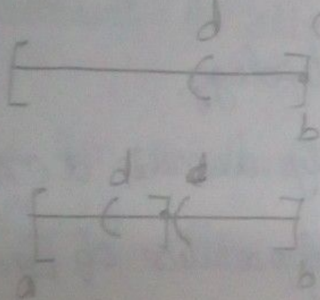

fig-24.1

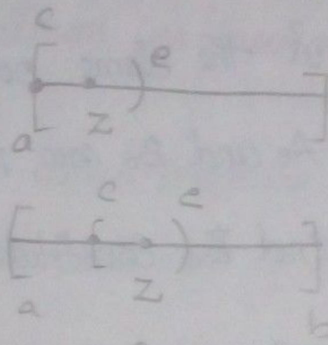

fig-24.2

Case: ii:

Suppose that $c \in A_0$. Then $c \neq b$, so either $c = a$ or $a < c < b$.

Because A_0 is open in $[a, b]$, there must be some interval of the form $[c, e)$ contained in A_0. (fig 24.2) Because of order property (2) of the linear continuum L,

We can choose a point z of L such that $c < z < e$. Then $z \in A_0$.

Contrary to the fact that c is an upper bound for A_0.

Corollary 24.2: The real line $\mathbb{R}$ is connected and so are intervals and rays in $\mathbb{R}$.

As an application, we P.T intermediate value thm of Calculus, suitably generalised.

Thm 24.3 INTERMEDIATE VALUE THEOREM.

Let $f: X \to Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a and b two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a points c of X such that $f(c) = r$.

The intermediate value theorem of calculus is the special case of this thm that occurs when we take X to be a closed interval in $\mathbb{R}$ and Y to be $\mathbb{R}$.

Proof:-

Assume the hypotheses of the thm.

The sets $A = f(X) \cap (-\infty, r)$ and

$B = f(X) \cap (r, +\infty)$ are disjoint,

and they are nonempty because one contains $f(a)$ and the other contains $f(b)$.

Each is open in $f(X)$, being the intersection of an open ray in Y with $f(X)$.

If there were no point c of X such that $f(c) = r$, Then $f(X)$ would be the union of the set A and B. Then A and B would constitute a separation of $f(X)$. Contradicting the fact that the image of a connected space under a continuous map is connected.

<u>Definition:</u>

Given points x and y of the space X, a path in X from x to y is a continuous map $f: [a,b] \to X$ of some closed interval in the real line X, such that $f(a) = x$ and $f(b) = y$.

A space X is said to be <u>path connected</u> if every pair of points of X can be joined by a path in X.

<u>Ex 3:</u>

Define the unit ball B^n in $\mathbb{R}^n$ by the eqn:-

$$B^n = \{x \mid \|x\| \leq 1\}.$$

where.

$$\|x\| = \|(x_1, \ldots x_n)\|$$

$$= (x_1^2 + \cdots + x_n^2)^{1/2}$$

The unit ball is path connected. given any two points x and y of B^n,

The straight-line path $f: [0,1] \to \mathbb{R}^n$ defined by $f(t) = (1-t)x + ty$.

lies in B^n.

For if x and y are in B^n and t is in $[0,1]$

$\| f(t) \| \leq (1-t)\|x\| + t\|y\|$.
≤ 1

A similar argument shows that every open ball $B_d(x, \epsilon)$ and every closed ball $\bar{B}_d(x, \epsilon)$ in $\mathbb{R}^n$ is path connected.

Ex: 4

Define Punctured euclidean space:-

Define Punctured euclidean space to be the space $\mathbb{R}^n - \{0\}$, where 0 is the origin in $\mathbb{R}^n$.

If $n > 1$, this space is path connected.

Given x and y different from 0, we can join x and y by the straight-line path between them if that path does not go through the origin.

otherwise, we can choose a point z not on the line joining x and y, and take the broken-line path from x to z, and then from z to y.

Ex: 5

Define the unit sphere S^{n-1} in $\mathbb{R}^n$ by the eqn.

$$S^{n-1} = \{x \mid \|x\| = 1\}.$$

If $n > 1$, it is path connected. For the map $g : \mathbb{R}^n - \{0\} \to S^{n-1}$ defined by $g(x) = x / \|x\|$ is continuous and surjective, and it is easy to S.T Continuous image of a path-connected space is Path connected.

25 Components and local connectedness:

Definition:-

Given X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing both x and y. The equivalence classes are called the Components (or) "Connected Components" of X.

Thm 25.1

The components of X are connected disjoint subspaces of X whose union is X, such that each non empty connected subspace of X intersects only one of them.

Proof:-

Being equivalence classes,

The Components of X are disjoint and their union is X. Each connected subspace A of X intersects only one of them.

For if A intersects the components C_1 and C_2 of X, say in points x_1 and x_2, respectively.

Then $x_1 \sim x_2$ by defn.

This cannot happen unless $C_1 = C_2$.

To show the component C is connected, choose a point x_0 of C. For each point x of C

$x_0 \sim x$, so there is a connected subspace A_x containing x_0 and x.

By the result just proved $A_x \subset C$.

$$\therefore \quad C = \bigcup_{x \in C} A_x.$$

Since the subspace A_x are connected and have the point x_0 in common, their union is connected.

Definition :-

We define another equivalence relation on the space X by defining $x \sim y$ if there is a path in X from x to y. The equivalence classes are called the path components of X.

Thm 25.2

The path components of X are path-connected disjoint subspaces of X whose union is X, such that each non empty path-connected subspace of X intersects only one of them.

Note that each component of spaces X is closed in X. Since the closure of a connected subspace of X is connected. If X has only finitely many components then each component is also open in X, since its complement is a finite union of closed sets.

But in general the components of X need not be open in X.

One can say even less about the path components of X, for they need be neither open nor closed in X.

Definition:

A space X is said to be locally connected at x if for every neighborhood U of x. There is a connected neighborhood V of x contained in U. If X is locally connected at each of its points, it is said simply to be locally connected.

IIIly

a space X is said to be locally path connected at x if for every neighborhood U of x, there is a path-connected neighborhood V of x contained in U.

If X is locally path connected at each of its points, Then it is said to be locally path connected.

Thm 25.3

A space X is locally connected if and only if for every open set U of X, each component of U is open in X.

Proof:-

Suppose that X is locally connected.

Let U be an open set in X, let C be a component of U

If x is a point of C, we can choose a connected neighborhood V of x such that $V \subset U$.

Since V is connected, it must lie entirely in the component of C of U.

Therefore, C is open in X.

Conversely,

Suppose that components of open sets in X are open. Given a point x of X and a neighborhood U of x.

Let C be the component of U containing x.

Now C is connected; since it is open in X by hypothesis. X is locally connected at x.

Thm 25.4

A space X is locally path connected iff for every open set U, of X, each path component of U is open in X.

The relation between path components and components is given in the following thm.

Thm 25.5

If X is a topological space, each path components of X lies in a component of X. If X is locally path connected, then the components and the path components of X are the same.

Proof:-

Let C be a component of X,

Let x be a point of C. Let P be the path component of x containing x. Since p is connected $p \subset c$.

To S.T if x is locally path connected, $p = c$.

Suppose that $p \subseteq c$. Let Q denote the union of all the path components of x that the are different from p and intersect c, each of them necessarily lies in c, so that $C = P \cup Q$.

Because x is locally path connected, each path component of x is open in x.

Therefore p (which is path component) and Q (which is union of path components) are open in x, so they constitute a separation of c.

This contradicts the fact that c is connected.