UNIT - IV

Compact spaces:

Definition :-

A collection $\mathcal{A}$ of subsets of a space X is said to cover X, or to be a covering of X, if the union of the elements of $\mathcal{A}$ is equal to X. It is called an open covering of if its elements are open subsets of X.

Defn:-

A space X is said to be Compact if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

✓ Lemma 26.1

Let Y be a subspace of X. Then Y is compact iff every covering of Y by sets open in X contains a finite subcollection covering Y.

Proof:

Suppose that Y is compact and $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ is a covering of Y by sets open in X.

Then the collection $\{A_\alpha \cap Y \mid \alpha \in J\}$ is covering of Y by sets open in Y.

hence a finite subcollection

$\{A_{\alpha_1} \cap Y, \dots A_{\alpha_n} \cap Y\}$. Covers Y.

Then $\{A_{\alpha_1}, \dots A_{\alpha_n}\}$ is a subcollection of $\mathcal{A}$ that covers Y.

Conversely,

Suppose the given condition holds,

We t.p Y compact.

Let $\mathcal{A}' = \{A'_\alpha\}$ be a covering of Y by sets open in Y.

For each α, choose a set A_α open in X such that

$$A'_\alpha = A_\alpha \cap Y.$$

The collection $\mathcal{A} = \{A_\alpha\}$ is a covering of Y by sets open in X.

By hypothesis, some finite subcollection $\{A_{\alpha_1}, \dots A_{\alpha_n}\}$ covers Y.

Then $\{A'_{\alpha_1}, \dots, A'_{\alpha_n}\}$ is a subcollection of $\mathcal{A}'$ that covers Y.

Thm 26.2

Every closed subspace of a compact space is compact.

Proof:

Let Y be a closed subspace of the compact space X.

Given a covering $\mathcal{A}$ of Y by sets open in X,

let us form an open covering $\mathcal{B}$ of X by adjoining to $\mathcal{A}$ the single open set $X-Y$, that is

$$\mathcal{B} = \mathcal{A} \cup \{X-Y\}.$$

Some finite subcollection of $\mathcal{B}$ covers X.

If this subcollection contains the set $X-Y$, discard $X-Y$; otherwise, leave the subcollection alone.

The resulting collection is a finite subcollection of $\mathcal{A}$ that covers Y.

Thm 26.3

Every compact subspace of a Hausdorff space is closed.

Proof:-

Let Y be a compact subspace of the Hausdorff space X.

W.P.T $X-Y$ is open, so that Y is closed.

Let x_0 be a point of $X-Y$.

We shown there is a neighborhood of x_0 that is disjoint from Y. For each point y of Y, let us choose disjoint neighborhoods U_y and V_y of the points x_0 and y, respectively, (using the Hausdorff condition).

The collection $\{V_y \mid y \in Y\}$ is a covering of Y by sets open in X;

$\therefore$ finitely many of them $V_{y_1}, \dots V_{y_n}$ cover Y.

The open set form $V = V_{y_1} \cup \dots \cup V_{y_n}$ contains Y, and it is disjoint from the open set

$$U = U_{y_1} \cap \dots \cap U_{y_n}$$

formed by taking the intersection of the corresponding neighborhoods of x_0. For if z is a point of V, then $z \in V_{y_i}$ for some i. hence $z \notin U_{y_i}$ and so $z \notin U$.

Then U is a neighborhood of x_0 disjoint from Y. as desired,

Lemma 26.4

If Y is a compact subspace of the Hausdorff space X and x_0 is not in Y, then there exist disjoint open sets U and V of X containing x_0 and Y, respectively.

Thm 26.5

The image of a compact space under a continuous map is compact.

proof:

Let $f: X \to Y$ be continuous, let X be compact.

Let $\mathcal{A}$ be a covering of the set $f(X)$ by sets open in Y.

The collection $\{f^{-1}(A) \mid A \in \mathcal{A}\}$ is a collection of sets covering X;

These set are open in X because f is continuous.

Hence finitely many of them, say

$$f^{-1}(A_1) \ldots f^{-1}(A_n),$$ cover X.

Then the sets $A_1, \ldots A_n$ cover $f(X)$.

Thm 26.6

Let $f: X \to Y$ be a bijective continuous function. If X is compact any Y is Hausdorff, then f is a homeomorphism.

proof:

W.P.T images of closed sets of X under f are closed in Y, this will prove continuity of the map f^{-1}.

If A is closed in X, then A is compact, by thm 26.2

"Every closed subspace of a compact space is compact."

Therefore, by proved $f(A)$ is compact. Since Y is Hausdorff, $f(A)$ is closed in Y, by thm 26.3

" Every compact subspace of a Hausdorff space is closed.

Thm 26.7

The product of finitely many compact spaces is compact.

proof:-

We shall prove that the product of two compact spaces is compact. The thm follows by induction for any finite product.

Step 1:

Suppose that ~~the~~ given spaces X and Y, with Y compact. Suppose that x_0 is a point of X, and N is an open set of $X \times Y$ containing the "slice" $x_0 \times Y$ of $X \times Y$.

W.P.T there is a neighborhood W of x_0 in X such that N contains the entire set $W \times Y$.

The set $W \times Y$ is often called a tube about $x_0 \times Y$.

First let us cover $x_0 \times Y$ by basis elements $U \times V$ (for the topology of $X \times Y$), lying in N.

The space $x_0 \times Y$ is compact, being homeomorphic to Y. Therefore, we cover $x_0 \times Y$ by finitely many such basis elements,

$$U_1 \times V_1, \ldots U_n \times V_n.$$

(We assume that each of the basis elements $U_i \times V_i$ actually intersects $x_0 \times Y$, since otherwise that basis element would be superflous; we could discard it from the finite collection and still have a covering of $x_0 \times Y$.)

Define $W = U_1 \cap \ldots \cap U_n$.

The set W is open, and it contains x_0 because each set $U_i \times V_i$ intersects $x_0 \times Y$.

~~we~~ the sets $U_i \times V_i$, which were chosen to cover the slice $x_0 \times Y$, actually cover the tube $W \times Y$.

Let $x \times y$ be a point of $W \times Y$.

Consider the point $x_0 \times y$ of the slice $x_0 \times Y$ having the same y co-ordinate as this point.

Now $x_0 \times y$ belongs to $U_i \times V_i$ for some i, so that $y \in V_i$. But $x \in U_j$ for every j (because $x \in W$).

Therefore, we have $x \times y \in U_i \times V_i$, as desired.

Since all the sets $U_i \times V_i$ lie in N, and since they cover $W \times Y$, the tube $W \times Y$ lies in N also. Fig 26.2

Step 2:

Now W.P.T Thm.

Let X and Y be compact spaces. Let A be an open

Covering of $X \times Y$. Given $x_0 \in X$. The slice $x_0 \times Y$ is compact and may therefore be covered by finitely many elements $A_1, \dots A_m$ of $\mathcal{A}$.

Their union $N = A_1 \cup \dots \cup A_m$ is an open set containing $x_0 \times Y$:

by Step 1, The open set N contains a tube $W \times Y$ about $x_0 \times Y$ where W is open in X. Then $W \times Y$ is covered by finitely many elements $A_1, \dots A_m$ of $\mathcal{A}$.

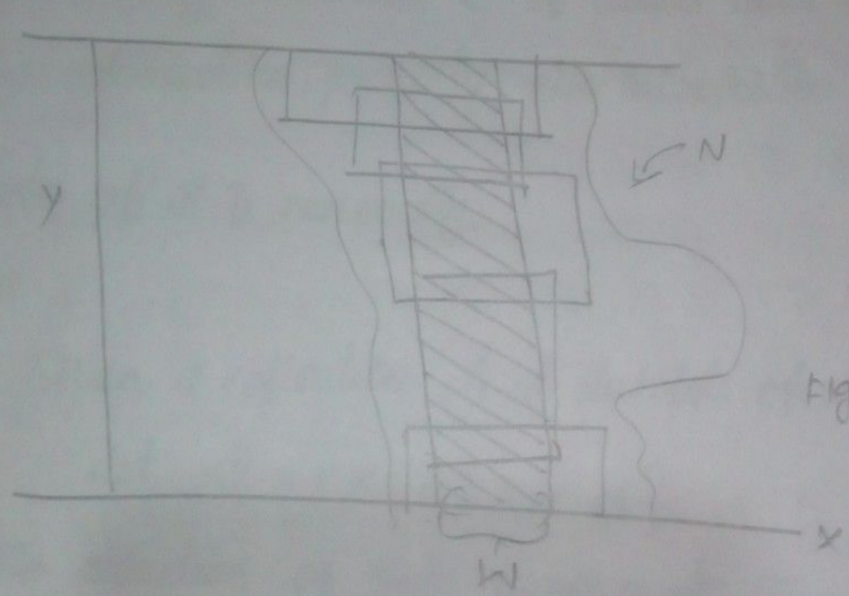

Fig 26.2

Thus, for each x in X, we can choose a neighborhood W_x of x such that the tube $W_x \times Y$ can be covered by finitely many elements of $\mathcal{A}$.

The collection of all the neighborhoods W_x is an open covering of X: therefore by compactness of X.

There exists a finite subcollection.

$$\{W_1, \dots W_k\}$$

Covering X. The union of the tubes $W_1 \times Y, \dots, W_k \times Y$ is all of $X \times Y$.

Since each may be covered by finitely many

Definition:

A collection $\mathcal{C}$ of subsets of X is said to have the finite intersection property if for every finite subcollection $\{C_1, \ldots, C_n\}$ of $\mathcal{C}$, the intersection $C_1 \cap \ldots \cap C_n$ is non empty.

Theorem 26.9:

Let X be a topological space. Then X is compact iff for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ of all the elements of $\mathcal{C}$ is nonempty.

proof:

Given a collection $\mathcal{A}$ of subsets of X,

let $\mathcal{C} = \{X - A \mid A \in \mathcal{A}\}$

be the collection of their complements. Then the following statements hold:-

(1) $\mathcal{A}$ is a collection of open sets if and if $\mathcal{C}$ is a collection of closed sets.

(2) The collection $\mathcal{A}$ covers X if and only if the intersection $\bigcap_{C \in \mathcal{C}} C$ of all the elements of $\mathcal{C}$ is empty

(3) The finite subcollection $\{A_1, \ldots A_n\}$ of $\mathcal{A}$ covers X iff The intersection of the corresponding elements $C_i = X - A_i$ of $\mathcal{C}$ is empty.

The first statement is trivial, while the second and third follow from DeMorgan's laws,

$$X - \left(\bigcup_{\alpha \in J} A_\alpha \right) = \bigcap_{\alpha \in J} (X - A_\alpha)$$

Taking the contrapositive (of the theorem) and then the complement

The Statement that X is compact is equivalent to saying. " Given any collection $\mathcal{A}$ of open subsets of X, if $\mathcal{A}$ covers X, then some finite subcollection of $\mathcal{A}$ covers X."

This Statement is equivalent to its contrapositive, Which is the following:

" Given any collection $\mathcal{A}$ of open sets, if no finite subcollection of $\mathcal{A}$ covers X, then $\mathcal{A}$ does not cover X."

Letting $\mathcal{C}$ be a as earlier, the collection $\{X-A \mid A \in \mathcal{A}\}$ and applying (1)–(3),

we see that this statement is in turn equivalent to the following:

" Given any collection $\mathcal{C}$ of closed sets, if every finite intersection of elements of $\mathcal{C}$ is nonempty, then the intersection of all the elements of $\mathcal{C}$ is nonempty."