Problem : 1

QP Show that $\{\frac{n}{n+1}\}$ is monotonically increasing sequence:

Solution:

Given $a_n = \{\frac{n}{n+1}\}$ to prove that monotonically increasing sequences.

$$a_n = \frac{n}{n+1} \quad \text{——— (1)}$$

$$a_{n+1} = \frac{n+1}{n+1+1} = \frac{n+1}{n+2}.$$

$$a_{n+1} - a_n = \left\{\frac{n+1}{n+2} - \frac{n}{n+1}\right\}$$

$$= \left\{\frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)}\right\}$$

$$= \left\{\frac{n^2+2n+1 - n^2 - 2n}{(n+2)(n+1)}\right\}$$

$$= \frac{1}{(n+2)(n+1)}.$$

$\therefore$ $a_{n+1} - a_n$ is a greater then (1) $\forall$ values of n hence it is monotonically increasing sequence.

Note:

A monotonic sequence always tend to limit finite or infinite.

Theorem: 1

A monotonic increasing sequence which is bounded above convergence to its least upper bound.

Proof:

Let $\{a_n\}$ be a monotonic increasing sequence which is bounded above.

Let 'K' be the least upper bounded of the sequence.

then $\boxed{a_n \leq K}$ $\forall n$ — (1).

Then let $\epsilon > 0$ be given,

$\therefore K-\epsilon < K$ and hence $K-\epsilon$ is not a upper bound of $\{a_n\}$. Hence there exist

$a_m \rightarrow : a_m > K-\epsilon$ — (2)

Now, since $\{a_n\}$ is monotonically increasing sequences.

$a_n \leq a_m$, $\forall n \geq m$ — (3)

Hence a_n is greater than $K-\epsilon$ for all $n \geq m$.

$a_n > K-\epsilon$, $\forall n \geq m$ — (4).

(ie) $\boxed{K-\epsilon < a_n}$ — (5)

from (1) & (5) we get

$K-\epsilon < a_n < K$, $\forall n \geq m$.

$|a_n - K| < \epsilon$.

$\{a_n\} \rightarrow K$.

Theorem: 2

Let $\{a_n\}$ be a monotonically increasing sequence which is not bounded above diverges to infinity.

Proof:

Let $K > 0$ be any real numbers.

since $\{a_n\}$ is not bounded above

there exist $m \in N, \rightarrow a_m > x$.

Also $a_n < a_m, \forall n \geq m$

$a_n > k, \forall n \geq m$

$\therefore \{a_n\} \rightarrow \infty$.

Theorem: 3

A monotonic decreasing sequence which is bounded below converges to its greatest lower bound.

Proof:

Let $\{a_n\}$ be a monotonic decreasing sequence which is bounded below.

Let k be the greatest lower bounded of the sequence.

then $a_n \geq k, \forall n$ — ①.

Then let $\epsilon > 0$. be given.

$k < k+\epsilon$ and hence $k+\epsilon$ is not a lower bound of $\{a_n\}$.

Hence there exist $a_m \rightarrow: k+\epsilon > a_m$ — ②.

Now since $\{a_n\}$ is monotonic decreasing.

$\therefore a_n \leq a_m, \forall n \geq m$ — ③

Hence $a_n \leq a_m < k+\epsilon, \forall n \geq m$ — ④.

$k+\epsilon > a_n \geq k, \forall n \geq m$ — ⑤.

[from ① & ⑤]

$\therefore |a_n - k| < \epsilon, \forall n \geq m$

$\therefore \{a_n\} \to k.$

Theorem: 4.

Let $\{a_n\}$ be a monotonic decreasing sequence which is not bounded below diverges to $-\infty$.

Proof:

Let $k > 0$ be any real numbers.

Since $\{a_n\}$ is not bounded below there exist $m \in N$ such that $a_m < k$.

Also $a_n \geq a_m, \quad \forall n \geq m$

$a_n \leq k, \quad \forall n \geq m$

$\therefore \{a_n\} \to -\infty.$

Note:

The above theorem shows that monotonic sequence either converges to diverges.

Problems: I

1. If $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}$ Show that $\{a_n\} \to$ limit. (or) S.T $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}$ is a monotonic increasing sequence.

Soln:

Let $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n}$

$a_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n+2}.$

$$(a_{n+1} - a_n) = \left(\frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n+2}\right) -$$
$$\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right).$$

$$= \left(\frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}\right) -$$
$$\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right)$$

$$= \left[\frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}\right.$$
$$\left. - \frac{1}{n+2} - \cdots - \frac{1}{2n}\right].$$

$$= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}.$$

$$= \frac{1}{2n+1} + \frac{1}{2(n+1)} - \frac{1}{n+1}.$$

$$= \frac{2(n+1) + (2n+1) - 2(2n+1)}{2(2n+1)(n+1)}.$$

$$= \frac{2n+2+2n+1-4n-2}{2(n+1)(2n+1)}$$

$$= \frac{1}{2(n+1)(2n+1)}.$$

$a_{n+1} - a_n > 0$ $\qquad$ $\le a_{n+1} - a_n$

$a_{n+1} > a_n.$

Hence $\{a_n\}$ is a monotonic increasing sequence.

$$a_n < \frac{1}{n+1} + \frac{1}{n+1} + \cdots + \frac{1}{n+1}.$$

$$< \frac{n}{n+1} \qquad n \to \infty$$

$$< \frac{n(1)}{n\left(1+\frac{1}{n}\right)} < \frac{1}{1+1/\infty} = \frac{1}{1+0} = \frac{1}{1} = 1.$$

$a_n < 1.$

Hence $\{a_n\}$ is a bounded monotonic increasing sequence and so it tends to a limit.

2 Find the limit of the sequence $\{a_n\}$ where

1 $a_n = [1+\frac{1}{n}]^n$.

Q.P

Soln:

From the Binomial theorem when n is positive integer. we know that.

$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)}{3!}x^3 \cdots$

$$[1+\frac{1}{n}]^n = 1 + \frac{n \cdot 1/n}{1!} + \frac{n(n-1)}{2!}(\frac{1}{n})^2 + \frac{n(n-1)(n-2)}{3!}(1/n)^3$$

$$+ \cdots + \frac{n(n-1)(n-2)\cdots n(n-1)}{n!}(1/n)^n$$

$$= 1 + \frac{1}{1!} + \frac{n^2(1-1/n)}{2!}\frac{1}{n^2} + \frac{n^3(1-\frac{1}{n})(1-\frac{2}{n})}{3!}\frac{1}{n^3}$$

$$+ \cdots + \frac{1}{n!}(1-\frac{1}{n}) \cdots (1-\frac{(n-1)}{n})$$

This expressions contains $(n+1)$ terms.

As, n increases, the number of terms is increases as also every one of the terms after the second term.

Hence $\{a_n\}$ is a monotonic increasing sequence.

To show that it is bounded above,

$$(1+\frac{1}{n})^n < 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \quad [\because \frac{1-x^m}{1-x}]$$

$$< 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} \cdots$$

$$< 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}$$

$< 1 + \frac{1-(1/2)^n}{1-1/2} < 1 + \frac{1-(1/2)^n}{1/2}$

$< 1 + \frac{1}{1/2} - \frac{1/2^n}{1/2}$

$< 1 + 2 - \frac{1}{2^n} \times 2$

$< 3 - \left(\frac{1}{2^{n-1}}\right)$

$< 3 - \left(\frac{1}{2^{n-1}}\right)$

$a_n < 3 \quad \forall n$

$\therefore$ The $\{a_n\} = \{1 + 1/n\}^n$ is montonically increase and bounded above.

$\therefore$ It is convergence.

$\therefore \lim_{n \to \infty} a_n = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$

$\therefore \lim_{n \to \infty} a_n = e$

so $2 < e < 3$.

3. If $a_{n+1} = \frac{1}{2}(a_n + b_n)$, $b_{n+1} = \sqrt{(a_n b_n)}$ S.T the sequences $\{a_n\}$ and $\{b_n\}$ converge to a common limit.

Proof:

If A, G be the Arithmetic mean and geometric mean respectively 'a' and 'b' $(a > b)$

$A.M > G.M$

then $b < G < A < a$. $[\because g_n]$

$b_n < b_{n+1} < a_{n+1} < a_n$

Hence in this case $b_n \leq b_{n+1} \leq a_{n+1} \leq a_n$

Now $a_{n+1} < a_n$.

Hence $\{a_n\}$ is a monotonic decreasing sequence and b_1 is one of its lower bounds

$\therefore \{a_n\} \to$ limit l_1 — (1)

$b_n < b_{n+1}$

Similarly $\{b_n\}$ is a monotonic increasing sequence and a_1 is one of its upper bounds.

$\therefore \{b_n\} \to$ limit l_2 — (2).

Now $a_{n+1} = \frac{1}{2}(a_n + b_n)$

$b_{n+1} = \sqrt{n}$.

Taking limits on both sides, we get.

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n + b_n)$$

$$l_1 = \pm \frac{1}{2}(l_1 + l_2)$$

$$l_1 = \frac{l_1 + l_2}{2}$$

$$l_1 - l_1/2 = \frac{l_2}{2}.$$

$$\frac{l_1}{2} = \frac{l_2}{2},$$

$$l_1 = l_2.$$

Hence proved.

4. If $a_{n+2} = \sqrt{a_{n+1}\, a_n}$ and $a_n > 0$ s.t $\{a_{(2n-1)}\}$ and $\{a_{2n}\}$ are both monotonic, one increasing and the other decreasing and the $\{a_n\}$ tends to $(a_1 a_2^2)^{1/3}$.

Proof:

Let $a_1 > a_2$

$\therefore a_3 = \sqrt{a_1 . a_2}$ $\therefore a_3$ lies between a_1 & a_2

$\therefore a_4 = \sqrt{a_2 . a_3}$ $\therefore a_4$ lies between a_2 & a_3.

From this we see that.

$a_2, a_4, a_6 \ldots$ is monotonic increasing sequence

$a_1, a_3, a_5 \ldots$ is monotonic decreasing sequence

$n=1$ $a_3^2 = a_1 . a_2$.

dividing by a_2^2.

$$\frac{a_3^2}{a_2^2} = \frac{a_1}{a_2}.$$

$$\frac{a_3}{a_2} = \left(\frac{a_1}{a_2}\right)^{1/2} \quad \text{—} \quad (1).$$

$$\left(\frac{a_4}{a_2}\right)^2 = \frac{a_3 . a_2}{a_2^2} = \frac{a_3}{a_2}.$$

$$\left(\frac{a_4}{a_2}\right)^2 = \left(\frac{a_1}{a_2}\right)^{1/2} \quad [\because \text{using } (1)].$$

$$\frac{a_4}{a_2} = \left[\left(\frac{a_1}{a_2}\right)^{1/2}\right]^{1/2}.$$

$$\frac{a_5^2}{a_2^2} = \frac{a_3 . a_4}{a_2^2}$$

$$\frac{a_5^2}{a_2^2} = \frac{a_3}{a_2} . \frac{a_4}{a_2}.$$

$$= \left[\frac{a_1}{a_2}\right]^{1/2} \left[\frac{a_1}{a_2}\right]^{1/4}$$

$$\frac{a_5^2}{a_2^2} = \left[\frac{a_1}{a_2}\right]^{3/4}.$$

$$\left[\frac{a_5}{a_2}\right]^2 = \left[\frac{a_1}{a_2}\right]^{3/4}.$$

we get $\frac{a_{n+2}}{a_2} = \left(\frac{a_1}{a_2}\right)^{u_n}$.

where u_n is the n^{th} term of the (series) sequence $\overset{u_1}{1/2} \cdot \overset{u_2}{1/4} \cdot \overset{u_3}{3/8} \cdot \overset{u_4}{5/16} \cdots$ here

$u_n = 1/2\,[u_{n-1} + u_{n-2}]$.

$u_n - u_{n-1} = \left\{ \frac{u_{n-1}}{2} + \frac{u_{n-2}}{2} - u_{n-1} \right\}$ (sub. u_{n-1})

$u_n - u_{n-1} = \frac{u_{n-1}}{2} + \frac{u_{n-2}}{2} - u_{n-1}$

$u_n - u_{n-1} = \frac{u_{n-1}}{2} + \frac{u_{n-2}}{2} - u_{n-1} \Rightarrow \frac{u_{n-1} + u_{n-2} - 2u_{n-1}}{2}$

$\frac{u_{n-2} - u_{n-1}}{2}$

$u_n - u_{n-1} = \frac{-1}{2}(u_{n-1} - u_{n-2})$ — (1) $-\frac{1}{2}(u_{n-1} - u_{n-2})$

$u_{n-1} - u_{n-2} = -1/2\,(u_{n-2} - u_{n-3})$ — (2)

$u_{n-2} - u_{n-3} = \frac{-1}{2}(u_{n-3} - u_{n-4})$ — (3)

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$n=3$ in (1)

$u_3 - u_2 = -1/2\,(u_2 - u_1)$ — (4)

Substitute $n, n-1, \ldots$ (2) in (1). we get,

$u_n - u_{n-1} = (-1/2)^{n-2}\,(u_2 - u_1)$

$= (-1/2)^{n-2}\,(-1/2)^2$

$= \frac{(-1)^{n-2}}{2^{n-2}}\left(\frac{-1}{2^2}\right)$

$u_n - u_{n-1} = \frac{(-1)^{n-1}}{2^n}$

$u_{n-1} - u_{n-2} = \frac{(-1)^{n-2}}{2^{n-1}}$

$u_{n-2} - u_{n-3} = \frac{(-1)^{n-3}}{2^{n-2}}$

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$$u_3 - u_2 = \frac{(-1)^{n-3}}{2^{n-2}}$$

$$u_n - u_1 = \frac{-1}{2^2} + \frac{1}{2^3} + \dots + \frac{(-1)^{n-1}}{2^n}$$

$$u_n = \frac{1}{2}\left[1 - \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{(-1)^{n-1}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1-(-1/2)^n}{1-(-1/2)}\right].$$

$$u_n = \frac{1}{3}\left[1 + (\tfrac{-1}{2})^n\right].$$

$$u_n = \frac{1}{3}\left[1 + (\tfrac{-1}{2})^{n+1}\right]$$

$$\lim_{n\to\infty} u_n = \frac{1}{3}$$

Now $\frac{a_{n+2}}{a_2} = \left[\frac{a_1}{a_2}\right]^{u_n}.$

$$\lim_{n\to\infty} \frac{a_{n+2}}{a_2} = \left[\frac{a_1}{a_2}\right]^{\lim_{n\to\infty} u_n}.$$

$$\frac{a_{n+2}}{a_2} = \left[\frac{a_1}{a_2}\right]^{1/3}$$

$$\lim_{n\to\infty} a_{n+2} = a_2 \cdot \frac{a_1^{1/3}}{a_2^{1/3}} = a_2^{2/3} \cdot a_1^{1/3}$$

$$\lim_{n\to\infty} a_{n+2} = (a_1 \cdot a_2^2)^{1/3}.$$

Hence proved.

Exercise

1) Examine the behaviour of the sequence $\{a_n\}$ where a_n is equal to.

i) $a_n = \frac{(-1)^n n}{2n-1}$

$\{a_n\} = \{-1, -2/3, -3/5, \frac{4}{-1}, \dots\}.$

This sequence is bounded below by -1 and is not bounded above 1.

ii) $a_n = n + (-1)^n \cdot 2n$

$\{a_n\} = \{-1, 6, -3 \dots\}$

$\{a_n\}$ is not bounded above and it is bounded below -1.

iii) $a_n = n\{1 + (-1)^n\}$

$\{a_n\} = \{0, 4, 0, 8, 0 \dots\}$

This sequence $\{a_n\}$ is bounded below by zero and is not bounded above.

iv) $a_n = 2 + (-1)^n$

$\{a_n\} = \{1, 3, 1, 3, 1, -3 \dots\}$

This sequence $\{a_n\}$ is bounded above by 3 is bounded below by $+1$.

v) $a_n = n + \frac{(-1)^n}{n^2}$

$\{a_n\} = \{0, 2, +\frac{1}{4}, 3 - \frac{1}{9}, 4 + \frac{1}{16}, \dots\}$

This sequence $\{a_n\}$ is bounded below by 0 and it is bounded above.

vi) $a_n = 2 + \frac{(-1)^n}{n}$

$\{a_n\} = \{2, -1, 2 + \frac{1}{2}, 2 - \frac{1}{3}, 2 + \frac{1}{4} \dots\}$

This sequence $\{a_n\}$ is bounded below by one and is bounded above by 3.

vii) $a_n = (n+1)^{1/n} - (n-1)^{1/n}$

$\{a_n\} = \{2, -0, 3^{1/2}, -1^{1/2}, 4^{1/3}, -2^{1/3} \ldots\}$

This sequence $\{a_n\}$ is bounded above by 2 and is bounded below by 0.

viii) $a_n = (n^3+1)^{1/3} - n$

$\{a_n\} = \{2^{1/3}-1, 9^{1/3}-2, 28^{1/3}-3 \ldots\}$

$[(n^3+1)^{1/3} - n < 1]$

$(n^3+1)^{1/3} < (n+1)$

$(n^3+1) < (n+1)^3$

This sequence $\{a_n\}$ is bounded above by one and is bounded below by 0.

2) Examine the following sequence for convergences

i) $n \cdot \tan(1/n)$

$$\lim_{n\to\infty} n \tan(1/n) = \lim_{n\to\infty} \frac{\tan(1/n)}{1/n} = 1$$

$[\because n\to\infty, 1/n \to 0]$

This sequence is convergence,

$\frac{\tan\theta}{\theta}$ as $\theta \to 0$.

$$\lim_{n\to\infty} \frac{\tan\theta}{\theta} = 1.$$

The given series is convergent to 1.

ii) $\frac{an^2+bn+c}{pn^2+qn+r}$

$$\lim_{n\to\infty} \frac{n^2(a+b/n+c/n^2)}{n^2(p+q/n+r/n^2)} = \frac{a}{p} \quad [\because \lim_{n\to\infty} \frac{1}{n} = 0]$$

This sequence is converges to a/p.

3. P.T $\frac{n!}{n^n}$ is convergence

Proof:

Let $a_n = \frac{n!}{n^n}$ $(a_n > a_{n+1})$

Then $\frac{a_n}{a_{n+1}} = \frac{n!}{n^n} \cdot \frac{(n+1)^{n+1}}{(n+1)!}$

$= \left(\frac{n+1}{n}\right)^n > 1.$

$\therefore a_n > a_{n+1}, \forall n \in N.$

$\{a_n\}$ is a monotonically decreasing sequence.

Also $a_n > 0, \forall n \in N.$

$\{a_n\}$ is bounded below

$\therefore \{a_n\}$ is convergence.

4. S.T $\lim_{n\to\infty} \frac{1}{n}\left[1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right] = 0.$

[Application of Cauchy's 1st limit theorem].

Soln:

Let $a_n = \frac{1}{n}.$

W.K.T $\{a_n\} \to 0.$

Hence by Cauchy's 1st limit theorem we get,

$$\left\{\frac{a_1+a_2+\cdots+a_n}{n}\right\} \to 0.$$

$$\left\{\frac{1}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\right\} \to 0.$$

5. Let $a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}$ In the $\{a_n\}$,

S.T $\lim_{n \to \infty} \{a_n\}$ exists & lies between 2 & 3.

Soln:

Given $a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}$

$a_{n+1} = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} + \frac{1}{(n+1)!}$

$a_{n+1} - a_n = [1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} + \frac{1}{(n+1)!} - 1 - \frac{1}{1!} - \frac{1}{2!} - \dots - \frac{1}{n!}]$

$a_{n+1} - a_n = \frac{1}{(n+1)!} > 0.$

$\therefore a_{n+1} > a_n.$

Hence $\{a_n\}$ is a monotonic increasing sequence.

Again. $a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}$

$= 1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \dots + \frac{1}{1\cdot 2\cdot 3 \cdots n}$

$< 1 + 1 + \frac{1}{2} + \frac{1}{1\cdot 2\cdot 2} + \dots + \frac{1}{1\cdot 2\cdot 2 \cdots n}.$

$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}$

$= 1 + [\frac{1 - (1/2)^n}{1 - 1/2}]^2 = 1 + 2[1 - (1/2)^n]$

$= 1 + 2 - 2[1/2]^n$

$= 3 - \frac{2}{2^n} = 3 - \frac{1}{2^{n-1}}.$

$a_n < 3.$

$\{a_n\}$ is bounded above the given sequence is $\{a_n\}$ is monotonically increasing & bounded above.

$\Rightarrow \lim_{n \to \infty} a_n$ exists also

$2 < a_n < 3$

$2 < \lim_{n \to \infty} < 3.$

Hence the result.