UNIT - II

Infinite series:

An expression is of the form $u_1+u_2+u_3+\cdots+u_n\cdots$ in which every term is followed by another term is called an infinite series. The series is denoted by $\sum_{r=1}^{\infty} u_r$

Ex: $1\cdot2\cdot3+2\cdot3\cdot4+3\cdot4\cdot5+4\cdot5\cdot6+\cdots$

Note:

As $n\to\infty$, there are four distinct possiblities for s_n.

* s_n may tend to a finite limit
* s_n may tend to a infinity (∞)
* s_n may tend to a $(-\infty)$
* s_n may tend to more than one limit.

Definitions:

Sum to infinity:

If s_n tends to a finite limit (say s) then the series is said to be convergent and s is called sum to infinite.

Example:

Consider the series $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$

Soln:

$$s_n = \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}$$

$$= \frac{(2^n-1)/2^n}{1/2} = \frac{(2^n-1)}{2^n} \times \frac{2}{1}$$

$$S_n = \frac{2(2^n-1)}{2^n}$$

$$S_n = 2 - \frac{1}{2^{n-1}}$$

As $n \to \infty$, $\frac{1}{2^{n-1}} \to 0$

$$S_n = 2 - 0$$

$$S_n = 2$$

The series is convergence.

ii) If S_n tends to a infinity or to minus infinity the series is said to be divergent.

Ex: $1+2+3+\dots = \infty$

$1+1+1 \dots$ = (divergent)

iii) If S_n tends to a more than one limit the series is said to be oscillate.

Ex: $1-1+1 \dots = 0$.

1. Test the convergence of the geometric series $1+x+x^2+x^3+\dots$

5M

QP

10m Soln:

Given $1+x+x^2+x^3+\dots$

we know that

$$1+x+x^2+x^3+\dots = \frac{1-x^n}{1-x} \quad ; x \neq 1$$

Also it $|x| < 1$

$-1 < x < 1$, $x^n \to 0$ as $n \to \infty$

$$s_n = \frac{1}{1-x} \text{ as } n \to \infty$$

Given geometric series can be consider as follows:

$|x| < 1 \to$ the series is convergent.

$x = 1 \to$ The series is divergent.

$x = -1 \to$ the series is oscillate.

$x > 1 \to$ the series is divergent to infinity.

$x < -1 \to$ The series is ~~to~~ divergent to $+\infty$ (or) $-\infty$ according as n is odd (or) even.

Some general theorem's concerning infinite series:

Theorem 1:

Statement:

If $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ is convergent and has the sum s, then $u_{m+1} + u_{m+2} + \cdots$ is convergent and has the sum $s - (u_1 + u_2 + \cdots + u_m)$ where 'm' is any positive integer.

Proof:

$$\lim_{n \to \infty} (u_{m+1} + u_{m+2} + \cdots + u_{m+n})$$

$$= \lim_{n \to \infty} \{ (u_1 + u_2 + \cdots + u_m) + (u_{m+1} + \cdots + u_{m+n}) - (u_1 + u_2 + \cdots + u_m) \}$$

$$= \lim_{n\to\infty} (u_1+u_2+\cdots+u_{m+n}) - (u_1+u_2+\cdots+u_m)$$

$$= \lim_{n\to\infty} (u_1+u_2+\cdots+u_{m+n}) - \lim_{n\to\infty} (u_1+u_2+\cdots+u_m)$$

$$= S - (u_1+u_2+\cdots+u_m)$$

Similarly, if $u_1+u_2+\cdots+u_n+\cdots$ is diverges, then $u_{m+1}+u_{m+2}+\cdots+u_{m+n}$ is divergent [where m is any given positive integer].

Theorem: 2

If $u_1+u_2+\cdots$ is convergent and has the sum 's', then $ku_1+ku_2+\cdots+$ is convergent and has the sum ks.

Proof:

Given $\lim_{n\to\infty} u_1+u_2+\cdots+u_n = s$ — (1)

$$\lim_{n\to\infty} (ku_1+ku_2+\cdots+ku_n) = \lim_{n\to\infty} k(u_1+u_2+\cdots+u_n)$$

$$= k \lim_{n\to\infty} (u_1+u_2+\cdots+u_n)$$

$$= k.s. \quad \text{[From (1)]}$$

$\therefore ku_1+ku_2+\cdots+ku_n+\cdots$ converges to ks.

Theorem: 3

Q.P If $u_1+u_2+\cdots+u_n+\cdots$ and $v_1+v_2+\cdots+v_n$ are both convergent, the series $\Sigma(u_n+v_n)$ is convergent and its sum is the sum of the two series.

Proof:

Let the sum of the two series be s and t respectively.

Given that,

let $\lim_{n\to\infty} (u_1+u_2+\dots+u_n) = s$
$\lim_{n\to\infty} (v_1+v_2+\dots+v_n) = t$ — ①

$\lim_{n\to\infty} \{(u_1+v_1) + (u_2+v_2) + (u_3+v_3) + \dots + (u_n+v_n)\}$

$= \lim_{n\to\infty} \{(u_1+u_2+u_3+\dots+u_n) + (v_1+v_2+\dots+v_n)\}$

$= \lim_{n\to\infty} u_1+u_2+u_3+\dots+u_n + \lim_{n\to\infty} v_1+v_2+v_3+\dots+v_n$

$= s+t$ ($\because$ from ①)

Hence proved.

Series of positive terms:-

Theorem: 1

A series of positive terms cannot oscillate, It is either convergent or divergent.

Proof:

Since all the terms are positive, S_n steadily increases as n increases.

$\therefore$ It tends to a finite limit or to infinity. Hence the series cannot oscillate.

If $S_n < k$ for all values of n,

$\lim_{n \to \infty} S_n$ exists and is equal to k or is less than k.

Then the series is convergent.

Theorem: 2

If $u_1 + u_2 + \cdots + u_n$ is convergent, then $\lim_{n \to \infty} u_n = 0$.

Proof:

Given the series is convergent.

(ie), $\lim_{n \to \infty} (u_1 + u_2 + \cdots + u_n)$ is finite.

Let $S = \lim_{n \to \infty} (u_1 + u_2 + \cdots + u_n) \longrightarrow (1)$

Now, $u_n = (u_1 + u_2 + \cdots + u_n) - (u_1 + u_2 + \cdots + u_{n-1})$.

Take limit on both sides as $n \to \infty$.

$\lim_{n \to \infty} u_n = \lim_{n \to \infty} (u_1 + u_2 + \cdots + u_n) - \lim_{n \to \infty} (u_1 + u_2 + \cdots + u_{n-1})$

$\lim_{n \to \infty} u_n = S - S$

$\lim_{n \to \infty} u_n = 0$.

The converse is not true. That is when $u_n \to 0$ as $n \to \infty$, we cannot definitely say that $\sum u_n$ is a convergent series. This is evident from the following example.

Example:

$\sum (1/n) = 1 + 1/2 + 1/3 + \cdots$

Solution:

$1 + 1/2 \cancel{>} 1/2$.

$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{2}{4}$.

$\frac{1}{5} + \cdots + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8}$

$\frac{1}{9} + \cdots + \frac{1}{16} > \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \cdots + \frac{1}{16} = \frac{8}{16} = \frac{1}{2}$

$1 + \cdots + \frac{1}{2^n} > \frac{n}{1/2} = 2n$.

$(1+\frac{1}{2}) + (\frac{1}{3}+\frac{1}{4}) + (\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots) > 1+\frac{1}{2}+\frac{1}{2}\cdots$

The sum of 2^n terms $> 1+\frac{n}{2}$.

As $n \to \infty$, $S_n \to \infty$, the series $\Sigma(\frac{1}{n})$

But $\frac{1}{n} \to 0$.

Note:

i) If the limits of the n^{th} term is not zero, then we can see that the series is divergent.

ii) If the n^{th} term to zero from this alone the divergence see the convergence see the series cannot be determined.

Example:

$\Sigma(n^{-1/n})$

$\Sigma\left(\frac{n^2-1}{n^2+1}\right)$

$\Sigma \frac{n}{1+2^{-n}}$

n^{th} terms $n^{-1/n}$ for divergent. Since the limit of the terms each case is not zero.