UNIT - IV

Theorem:

Q? Cauchy's condensation test:

Let if $f(n)$ is positive for all positive integral value of n and continuosly diminish as n is increases and if 'a' be a positive integer than the two infinite series. $f(1) + f(2) + f(3) + \cdots + f(n)$

$af(a) + a^2 f(a^2) + a^3 f(a^3) + \cdots a^n f(a^n)$ are both convergent or divergent.

Proof:

Let us group the terms of $\Sigma f(n)$ as follows:

$$f(1) + f(2) + f(3) + \cdots f(n) + \cdots = f(1) + f(2) + f(3) + \cdots + f(a+1)$$
$$+ f(a+2) + f(a+3) + \cdots + f(a^2) \cdots$$
$$\cdots + f(a^{n-1}+1) + f(a^{n-1}+2) + \cdots + f(a^n)$$

Let v_n denote the terms of nth group

(ie) $f(a^{n-1}+1) + f(a^{n-2}+2) + \cdots + f(a^n)$

$\therefore$ The number of terms $a^n - a^{n-1}$

Since $f(n)$ is decreasing function.

$$(a^n - a^{n-1}) f(a^n) \le v_n \le (a^n - a^{n-1}) f(a^{n-1})$$

$$(a^n)(1 - 1/a) f(a^n) \le v_n \le a^n (1 - 1/a) f(a^n - 1)$$

$$\frac{a^n}{a}(a-1) f(a^n) \le v_n \le \frac{a^n}{a}(a-1) f(a^n - 1)$$

$\frac{a-1}{a}[a^n f(a^n)] \leq v_n \leq \frac{a-1}{a}[a^n f(a^{n-1})]$.

Now if $\sum a^n f(a^n)$ is finite.

So also is $\sum v_n$ is convergent.

$\sum v_n$ is the series $\sum f(n)$

$\therefore \sum f(n)$ is convergent.

Now if $\sum a^n f(a^n)$ is infinite so also is

$\sum v_n$ is divergent.

$\therefore \sum v_n$ is the series $\sum f(n)$.

$\therefore \sum f(n)$ is divergent.

1. Show that series $1+1/2+1/3+1/4+\cdots$ is divergent.

Soln:

Let $f(n) = 1/n$.

take $a=2$

By Cauchy's condensation test.

$\sum f(n)$ and $\sum a^n f(a^n)$

(ie) $\sum 1/n$ and $\sum 2^n (1/2^n)$

(ie) $\sum 1/n$ and $\sum 1$.

But $1+1+1+\cdots$ is divergent.

$\therefore \sum 1/n$ is divergent.

$\therefore$ The given series is divergent.

2. Discuss the convergence of the series $\sum \frac{1}{n(\log n)^p}$.

QP

Soln:

Let $f(n) = \frac{1}{n(\log n)^p}$.

By Cauchy's condensation test.

$\sum f(n)$ and $\sum a^n f(a^n)$

$$= \sum \frac{1}{n(\log n)^p} \text{ and } \sum \frac{a^n}{a^n(\log a^n)^p}$$

$$= \sum \frac{1}{n(\log n)^p} \text{ and } \sum \frac{1}{(\log a^n)^p}$$

$$= \sum \frac{1}{n(\log n)^p} \text{ and } \sum \frac{1}{(n \log a)^p}$$

$$= \sum \frac{1}{n(\log n)^p} \text{ and } \frac{1}{(\log a)^p} \sum \frac{1}{n^p}$$

Hence this series is convergent if $p>1$ and this series is divergent if $p \leq 1$.

3. Discuss the convergence of the series $\sum 1/n^k$.

Soln:

$$\text{let } f(n) = \frac{1}{n^k}$$

By Cauchy's condensation test.

$$\sum f(n) \text{ and } a^n \sum f(a^n)$$

Take $a = 2$

$$\sum 1/n^k \text{ and } \sum 2^n \left(\frac{1}{(2^n)^k}\right)$$

(ie) $\sum 1/n^k$ and $\frac{1}{2^{-n}} \cdot 2^{nk}$.

(ie) $\sum 1/n^k$ and $\frac{1}{2^{(k-1)n}}$.

$\sum \frac{1}{2^{(k-1)n}}$ which is a geometric series. by theorem '6'.

This series converges or diverges according as $k>1$ or $k \leq 1$.

$\sum 1/n^k$ is convergent if $k>1$.

$\sum 1/n^k$ is divergent if $k \leq 1$.

Cauchy's Root Test:

If $\sum_{n=1}^{\infty} u_n$ be a series of positive terms.

then the series is convergent or divergent according as $\lim_{n\to\infty} (u_n)^{1/n} < 1$ or > 1.

Proof:

Case (i)

$\lim_{n\to\infty} [u_n]^{1/n}$ be l where $l < 1$.

(ie) $\lim_{n\to\infty} [u_n]^{1/n} = l$, $l < 1$ and have we can

Choose ϵ positive and sufficiently small so that $l+\epsilon < 1$.

Since $\lim_{n\to\infty} [u_n]^{1/n} = l$.

We can find a natural number m, so large that u_n differ from l by less than ϵ so long as $n \geq m$

$\therefore [u_n]^{1/n} < l+\epsilon$.

$u_n < (l+\epsilon)^n$

Hence from and after m^{th} term the terms of the series Σu_n are less than these of the geometric series $\Sigma(l+\epsilon)^n$ which is convergent since $l+\epsilon < 1$.

$\therefore \Sigma u_n$ is convergent.

Case (ii)

Let $\lim [u_n]^{1/n}$ be l where $l > 1$ and hence we can choose a positive and sufficiently small so that $l-\epsilon > 1$

Now since $\lim_{n\to\infty} [u_n]^{1/n} = l$ we can find a natural number m so large that $[u_n]^{1/n}$ differ from 'l' by less than ϵ so long as $n \geq m$.

$\therefore [u_n]^{1/n} > l-\epsilon > 1.$

$\therefore u_n > (l-\epsilon)^n.$

$\Sigma(l-\epsilon)^n$ is divergent since $l-\epsilon > 1$.

Hence from and differ the m^{th} term the terms of the series and greater than the divergent series $\Sigma(l-\epsilon)^n$.

$\therefore \Sigma u_n$ is divergent.

Example:

1. Test for convergent the series $a+b+a^2+b^2+a^3+b^3+\ldots$

Soln:

$u_n = a^{\frac{n+1}{2}}$ when n is odd (or) $b^{n/2}$ when n is even.

$[u_n]^{1/n} = a^{\frac{n+1}{2n}}$ (or) $b^{1/2}$ according as n is odd or even.

$= a^{(1+1/n)1/2}$ or $b^{1/2}$ according as n is odd (or) even.

$\lim_{n\to\infty} u_n^{1/n} = a^{1/2}$ (or) $b^{1/2}$ according as n is odd or even.

$\therefore$ The series is convergent if $0 < a < 1$ and $0 < b < 1$ and divergent if $a \geq 1$ or $b \geq 1$.

2. Show that the series $\sum \frac{[(n+1)r]^n}{n^{n+1}}$ is convergent if $r<1$ and divergent if $r>1$.

Soln:

Let $u_n = \frac{[(n+1)r]^n}{n^{n+1}}$ [$\because$ by root test]

$$\therefore u_n^{1/n} = \frac{\{[(n+1)r]^n\}^{1/n}}{(n^{n+1})^{1/n}}$$

[$\lim_{n\to\infty} u_n^{1/n}$ is convergent $r<1$ if divergent $r>1$]

$$= \frac{(n+1)r}{n^{1+1/n}}$$

$$= \frac{(n+1)r}{n \cdot n^{1/n}}$$

$$= \frac{n(1+1/n)r}{n \cdot n^{1/n}} = \frac{(1+1/n)r}{\sqrt[n]{n}}$$

Taking limit as $n\to\infty$ on both sides.

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{(1+1/n)r}{\sqrt[n]{n}} \quad [\because \sqrt[n]{n}=1]$$

$$= \frac{(1)r}{1} = r.$$

If $r<1$, the given series is convergent.

$r>1$, the given series is divergent.

Let us consider the case when $r=1$.

$$u_n = \frac{[(n+1)r]^n}{[n^{n+1}]} = \frac{(n+1)^n}{n^{n+1}}$$

$$u_n = \frac{n^n[1+1/n]^n}{n^n \cdot n}$$

$$= \frac{[1+1/n]^n}{n}.$$

Let us choose $v_n = 1/n$.

$$\frac{u_n}{v_n} = \frac{(1+1/n)^n}{n} \times \frac{n}{1} = [1+1/n]^n.$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} (1+1/n)^n$$

$= e$ [e is finite number].

The series Σu_n, Σv_n behave alike.

$\therefore \Sigma v_n$ is divergent.

$\therefore \Sigma u_n$ is divergent

Hence the proof.

3. Examine the convergence of the series,

QP $\Sigma \frac{(n+1)(n+2)\cdots(n+n)}{n^n}$

Soln:

$$u_n = \frac{(n+1)(n+2)\cdots\cdots(n+n)}{n^n}.$$

$$= \frac{(n+1)}{n} \cdot \frac{n+2}{n} \cdots \frac{(n+n)}{n}.$$

$$= (1+1/n)(1+2/n)\cdots(1+n/n)$$

$$[u_n]^{1/n} = \{(1+1/n)(1+2/n)\cdots(1+n/n)\}^{1/n}.$$

$$\lim_{n\to\infty} [u_n]^{1/n} = \lim_{n\to\infty} [(1+1/n)(1+2/n)\cdots(1+n/n)]^{1/n}$$

let this limit be l.

$$\therefore \lim_{n\to\infty} [(1+1/n)(1+2/n)\cdots(1+n/n)]^{1/n} = l.$$

$$\log l = \lim_{n\to\infty} 1/n [\log(1+1/n) + \log(1+2/n) + \cdots + \log(1+n/n)]$$

$$= \lim_{n\to\infty} 1/n \sum_{n=1}^{\infty} \log(1+x/n)$$

$$= \int_0^1 \log(1+x)\,dx$$

$$= [x \log(1+x)]_0^1 - \int_0^1 \frac{x}{1+x}\,dx$$

$= \log 2 - [x - \log(1+x)]_0^1$

$= 2\log 2 - 1$

$= \log 2^2 - 1$

$\log l = \log 4 - \log_e e$

$l = 4/e$

since e lies between 2 & 3 $l > 1$.

Σu_n is divergent.

Absolute convergent series.

Definition:

The series Σu_n containing positive and negative terms is said to be absolutely convergent if the series formed by the numerical values of the term of Σu_n.

(ie) summation of $(u_n) = (\Sigma|u_n|)$ if convergent series.

Example:

The series Σu_n $1 - 1/2^2 + 1/3^2 - 1/4^2 + \cdots$ is absolutely convergent.

Conditionally convergent series or semi converges.

The series Σu_n containing positive and negative terms is said to be conditionally convergent (or) semi convergent.

If Σu_n is convergent and $\Sigma|u_n|$ is divergent.

Example

The series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\dots$ is conditionally convergent

Since $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots$ is divergent.

Theorem : 14

An absolutely convergent series is convergent

Proof:

Let $\sum u_n$ be the series.

Then by hypothesis $\sum |u_n|$ is convergent.

Now, $u_n + |u_n| = 2u_n$, if u_n is positive.

$= 0$, if u_n is negative.

$\therefore$ Every term of the series $\sum(u_n + |u_n|)$ is positive and is less than as equal to the corresponding terms of both the series is convergent.

Series whose terms are Alternatively positive and negative.

State and prove Lebiness theorem

theorem : 15.

If $u_1 - u_2 + u_3 - u_4 + \dots$ is a series of terms alternatively positive and negative and if $u_n > u_{n+1}$, $\forall n$ and $\lim_{n\to\infty} u_n = 0$ then the series is convergent.

Let S_{2n} denote the sum to $2n$ terms of the series.

then $S_{2n} = (u_1 - u_2) + (u_3 - u_4) + \dots + (u_{2n-1} - u_{2n})$

Since each bracket is positive, S_{2n} is steadly increases as n increases.

(ie) $S_2 < S_4 < S_6 < S_8 < \dots$ $< S_2 < S_4$

Without altering the given $S_4 < S_6$. order of the terms the sum, $S_6 < S_8$. S_{2n} may be written in the form.

$$S_{2n} = U_1 - (U_2 - U_3) - (U_4 - U_5) \dots (U_{2n-2} - U_{2n-1})$$

Since each bracket is positive.

$S_{2n} < U_1$.

$\lim_{n\to\infty} S_{2n}$ exist and equal ϵ or $l < U_1$ where $l < U_1$

(ie) $\lim_{n\to\infty} S_{2n} = l$.

But $\boxed{S_{2n+1} = S_{2n} + U_{2n+1}}$ and.

$\lim_{n\to\infty} U_{2n+1} = 0$. (alternatively +ve & -ve)

$\therefore \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} U_{2n+1}$.

$\lim_{n\to\infty} S_{2n+1} = l + 0$.

$\lim_{n\to\infty} S_{2n+1} = l$.

The series is convergent.