Binomial theorem for a rational index.

If n is a rational and $-1<x<1$ (ie) $|x|<1$ the sum of the series

$$1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)\cdots(n-r+1)}{r!}x^r+\cdots$$

is the real positive value of $(1+x)^n$.

Some important particular cases of the Binomial expansion:

$$(1-x)^{-1} = 1+(-1)(-x)+\frac{(-1)(-2)}{2!}(-x)^2+\cdots$$
$$= 1+x+x^2+x^3+\cdots$$

$$(1-x)^{-2} = 1+(-2)(-x)+\frac{(-2)(-3)}{2!}(-x)^2+\frac{(-2)(-3)(-4)}{3!}(-x)^3+\cdots$$
$$=1+2x+3x^2+4x^3+\cdots+(n+1)x^n+\cdots$$

$$(1-x)^{-3} = 1+(-3)(-x)+\frac{(-3)(-4)}{2!}(-x)^2+\frac{(-3)(-4)(-5)}{3!}(-x)^3\cdots$$
$$=1+3x+\frac{3\cdot4}{2!}x^2+\frac{4\cdot5}{3}x^3+\cdots+\frac{(n+1)(n+2)}{2}x^n+\cdots$$
$$=\frac{1}{2}\left[1\cdot2+2\cdot3x+3\cdot4x^2+4\cdot5x^3+\cdots+(n+1)(n+2)x^n+\cdots\right]$$

$$(1-x)^{-4} = 1+4x+\frac{4\cdot5}{2!}x^2+\frac{4\cdot5\cdot6}{3!}x^3+\cdots+\frac{(n+1)(n+2)(n+3)}{3!}x^n+\cdots$$
$$=\frac{1}{6}\left[1\cdot2\cdot3+2\cdot3\cdot4x+3\cdot4\cdot5x^2+4\cdot5\cdot6x^3+\cdots+(n+1)(n+2)(n+3)x^n+\cdots\right]$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots$$

$$(1-x)^{-1/2} = 1 + (+\tfrac{1}{2})(+x) + \frac{(+\frac{1}{2})(+\frac{1}{2}+1)}{2!}(+x)^2 + \frac{(+\frac{1}{2})(+\frac{1}{2}+1)(+\frac{1}{2}+2)}{3!}(+x)^3 + \cdots$$

$$= 1 + \frac{1}{2}x + \frac{(+\frac{1}{2})(\frac{3}{2})}{2\times 1}x^2 + \frac{(+\frac{1}{2})(\frac{3}{2})(\frac{5}{2})}{3\times 2}(+x)^3 + \cdots$$

$$= 1 + \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \cdots$$

$$(1-x)^{-1/3} = 1 + \frac{1}{3}x + \frac{1\cdot 4}{3\cdot 6}x^2 + \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}x^3 + \cdots$$

Example 1:

Find the general term in the expansion of $(1+x)^{2/3}$.

Soln:

$$(r+1)^n = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}\cdot x^r.$$

Thus $(r+1)^{th}$ term

$$= \frac{\frac{2}{3}(\frac{2}{3}-1)(\frac{2}{3}-2)\cdots(\frac{2}{3}-r+1)}{r!}x^r \quad [\because n = 2/3]$$

$$= \frac{\frac{2}{3}(\frac{-1}{3})(\frac{-4}{3})(\frac{-7}{3})\cdots(\frac{2-3r+3}{3})}{r!}x^r$$

$$= \frac{2(-1)(-4)(-7)\cdots(-3r+5)}{3^r\cdot(r!)}\cdot x^r$$

The number of factors in the numerator is r and $(r-1)$ of these are negative.

$\therefore$ The $(r+1)^{th}$ term $= (-1)^{r-1}\dfrac{2\cdot 1\cdot 4\cdot 7\cdots(3r-5)}{3^r\cdot(r!)}x^r$

Example 2:

Expand $[1+3x]^{5/2}$ given $|x| < 1/3$

Soln:

The function can be expanded in ascending powers of x if $|3x| < 1$, that is, if $|x| < \frac{1}{3}$.

$$(1+3x)^{5/2} = 1 + \frac{5}{2}(3x) + \frac{\frac{5}{2}\cdot\left(\frac{5}{2}-1\right)}{2!}(3x)^2 + \frac{\frac{5}{2}\left(\frac{5}{2}-1\right)\left(\frac{5}{2}-2\right)}{3!}(3x)^3 + \frac{\frac{5}{2}\left(\frac{5}{2}-1\right)\left(\frac{5}{2}-2\right)\left(\frac{5}{2}-3\right)}{4!}(3x)^4 + \dots$$

$$\left[\because (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots\right]$$

$$= 1 + 5\cdot\left(\frac{3x}{2}\right) + \frac{\frac{5}{2}\left(\frac{3}{2}\right)}{2!}(3x)^2 + \frac{\frac{5}{2}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{3!}(3x)^3 + \frac{\frac{5}{2}\cdot\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}{4!}(3x)^4 + \frac{\frac{5}{2}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{5!}(3x)^5$$
$$+ \frac{\frac{5}{2}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)}{6!}\cdot(3x)^6$$

$$= 1 + 5\cdot\left[\frac{3x}{2}\right] + \frac{5\cdot3}{2!}\left[\frac{3x}{2}\right]^2 + \frac{5\cdot3\cdot1}{3!}\left[\frac{3x}{2}\right]^3 + \frac{5\cdot3\cdot1\cdot(-1)}{4!}\left[\frac{3x}{2}\right]^4 + \frac{5\cdot3\cdot1(-1)(-3)}{5!}\left[\frac{3x}{2}\right]^5 + \frac{5\cdot3\cdot1\cdot(-1)(-3)(-5)}{6!}\left[\frac{3x}{2}\right]^6.$$

The terms which follow are alternately positive and negative the general term is,

$$= \frac{5\cdot3\cdot1\,(-1)(-3)(-5)\dots\left(\frac{5}{2}-r+1\right)}{r!}\left[\frac{3x}{2}\right]^r$$

$$= \frac{5.3.1.(-1)(-3)(-5)\cdots(5-2r+2)}{r!}\left[\frac{3x}{2}\right]^r$$

$$= \frac{5.3.1.(-1)(-3)(-5)\cdots(-2r+7)}{r!}\left[\frac{3x}{2}\right]^r$$

(ie) $(-1)^{r-3}\dfrac{5.3.1.1.3.5\cdots(2r-7)}{r!}\left[\frac{3x}{2}\right]^r ; r>3$

Example 3:

Expand $(a^3-2a^2x)^{5/3}$ in ascending powers of x.

Soln:

$$[a^3-2a^2x]^{5/3} = \left\{a^3\left(1-\frac{2x}{a}\right)\right\}^{5/3}$$

$$= (a^3)^{5/3}.\left(1-\frac{2x}{a}\right)^{5/3}$$

$$= a^5\left\{1+\frac{5}{3}\left(-\frac{2x}{a}\right)+\frac{\frac{5}{3}\cdot\left(\frac{5}{3}-1\right)}{2!}\left(-\frac{2x}{a}\right)^2+\frac{\frac{5}{3}\left(\frac{5}{3}-1\right)\left(\frac{5}{3}-2\right)}{3!}\left(-\frac{2x}{a}\right)^3+\cdots\right\}$$

$$= a^5\left\{1-\frac{5}{1!}\left(\frac{2x}{3a}\right)+\frac{\frac{5}{3}\left(\frac{2}{3}\right)}{2!}\left(-\frac{2x}{a}\right)^2+\frac{\frac{5}{3}\left(\frac{2}{3}\right)\left(\frac{-1}{3}\right)}{3!}\left(-\frac{2x}{a}\right)^3+\cdots\right\}$$

$$= a^5\left\{1-\frac{5}{1!}\left(\frac{2x}{3a}\right)+\frac{5.2}{2!}\left(\frac{2x}{3a}\right)^2+\frac{5.2.1}{3!}\left(\frac{2x}{3a}\right)^3+\cdots+\frac{5.2.1.4.7\cdots(5/3-r+1)}{r!}\left(\frac{2x}{3a}\right)^r+\cdots\right\}$$

$$= a^5\left\{1-\frac{5}{1!}\left(\frac{2x}{3a}\right)+\frac{5.2}{2!}\left(\frac{2x}{3a}\right)^2+\frac{5.2.1}{3!}\left(\frac{2x}{3a}\right)^3+\cdots+\frac{5.2.1.4.7\cdots(3r-8)}{r!}\left(\frac{2x}{3a}\right)^r+\cdots\right\}$$

2. Find the coefficient of x^n in the expansion of $\frac{x+1}{(x-1)^2(x-2)}$

Soln

Splitting the function into partial fractions, we have

$$\frac{x+1}{(x-1)^2(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-2)}$$

$$= \frac{A(x-1)+B(x-2)+C(x-1)^2}{(x-1)(x-1)^2(x-2)}$$

$A(x-1)+B(x-2)+C(x-1)^2 = x+1.$

Coeff of x^2: $C = 0$

Coeff of x: $A+B = 1.$

Constant: $-A-2B+C = 1 \Rightarrow -A-2B = 1.$

(1) $\Rightarrow$ $A+B = 1$

(2) $\Rightarrow$ $-A-2B = 1$

$-B = 2$. $\boxed{B=-2}$

Sub $B = 2$ in (1).

$A - 2 = 1$ $\boxed{A=3}$

$$\frac{x+1}{(x-1)^2(x-2)} = \frac{3}{(x-2)} - \frac{3}{(x-1)} - \frac{2}{(x-1)^2}$$

$$= +\frac{3}{2\left(\frac{x}{2}-\frac{2}{2}\right)} - \frac{3}{(x-1)} - \frac{2}{(x-1)^2}$$

$$= -\frac{3}{2(1-x/2)} - \frac{3}{(x-1)} - \frac{2}{(x-1)^2}$$

$$= -\frac{3}{2}(1-x/2)^{-1} + 3(1-x)^{-1} - 2(1-x)^{-2}$$

$$= -\frac{3}{2}\left\{1+\frac{x}{2}+\left(\frac{x}{2}\right)^2+\cdots\right\} + 3\left\{1+x+x^2+\cdots\right\}$$
$$-2\left\{1+2x+3x^2+\cdots (n+1)x^n+\cdots\right\}.$$

coefficient of $x^n = -\frac{3}{2}\cdot\frac{1}{2^n} + 3 - 2(n+1)$

$$= 1-2n-\frac{3}{2^{n+1}}$$

3. Prove that the coefficient of x^n in the expansion of $\frac{1+x}{(1+x^2)(1-x)^2}$ is $\frac{1}{2}\{(2n+3)+(-1)^{p-1}\}$, where $p=\frac{n}{2}$ or $\frac{n+1}{2}$ according as n is even or odd.

Soln:

We have $\frac{1+x}{(1+x^2)(1-x)^2} = \frac{\frac{1}{2}(x-1)}{1+x^2} + \frac{\frac{1}{2}}{1-x} + \frac{1}{(1-x)^2}$

$= \frac{1}{2}(x-1)(1+x^2)^{-1} + \frac{1}{2}(1-x)^{-1} + 1\cdot(1-x)^{-2}$

$= \frac{1}{2}(x-1)\{1-x^2+x^4\ldots\} + \frac{1}{2}\{1+x+x^2+\ldots\} + \{1+2x+3x^2+\ldots\}$

Let $n=2p$

If p is even, coeff of $x^{2p} = -\frac{1}{2}+\frac{1}{2}+(2p+1)$

$= 2p+1$

If p is odd, coeff of $x^{2p} = \frac{1}{2}+\frac{1}{2}+2p+1$

$= 2p+2$

∴ when $\frac{n}{2}=p$, coefficient of $x^n = n+1$ or $n+2$.

Let $n=2p-1$.

If p is even, coeff of $x^{2p-1} = -\frac{1}{2}+\frac{1}{2}+2p$

$= 2p$

and if p is odd, coeff of $x^{2p-1} = +\frac{1}{2}+\frac{1}{2}+2p$

$= 2p+1$

When $p=\frac{n+1}{2}$, coeff of $x^n = n+1$ or $n+2$

When p is even, coeff of $x^n = n+1$

$= \frac{1}{2}\{(2n+3)+(-1)^{p-1}\}$

When p is odd, coeff of $x^n = p+2$

$$= \frac{1}{2}\{(2n+3) + (-1)^{p-1}\}$$

Hence we get the result.

Formula:

1) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$

2) $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$

3) $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots$

4) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots$

Sum of coefficients:

If $f(x)$ can be expanded as an ascending series in x, we can find the sum of the first $(n+1)$ coefficients.

Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

$$\frac{f(x)}{1-x} = (a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots)(1 + x + x^2 + \cdots)$$

$\therefore$ coefficient of x^n in $\frac{f(x)}{1-x} = a_0 + a_1 + a_2 + a_3 + \cdots + a_n$

Thus to find the sum of the first $(n+1)$ coefficients in the expansion of $f(x)$, we have only to find the coefficients of x^n of the expansion of $f(x)/1-x$.

1. Find the sum of the coefficients of the first $(r+1)$ terms in the expansion of $(1-x)^{-3}$

The required result is the coefficient of x^r in the expansion $(1-x)^{-3}/1-x$

(ie) in the expansion of $(1-x)^{-4}$

(ie) in $1+4x+\frac{4.5}{2!}x^2+\frac{4.5.6}{3!}x^3+\dots\frac{(r+1)(r+2)(r+3)}{3!}x^r+\dots$

$\therefore$ Sum of the $(r+1)$ coefficients in the expansion of

$(1-x)^{-3}$ is $\frac{(r+1)(r+2)(r+3)}{6}$

2. If n is a positive integer and $\frac{(1+x)^n}{(1-x)^3}=a_0+a_1x+a_2x^2+\dots+a_nx^n+\dots$

S.T $a_0+a_1+a_2+\dots+a_{n-1}=\frac{1}{3}n(n+2)(n+7)2^{n-4}$

The sum required = coefficient of x^{n-1} in the expansion of

$(1+x)^n/(1-x)^3(1-x)$

$=$ " $(1+x)^n/(1-x)^4$

Now $(1+x)^n=\{2-(1-x)\}^n$

$=2^n-n.2^{n-1}(1-x)+\frac{n(n-1)}{2!}2^{n-2}.(1-x)^2$

$-\frac{n(n-1)(n-2)}{3!}2^{n-3}(1-x)^3+$ terms

involving powers of $(1-x)$, higher than third.

Hence $\frac{(1+x)^n}{(1-x)^4}=\frac{2^n}{(1-x)^4}-\frac{n.2^{n-1}}{(1-x)^3}+\frac{n.(n-1)2^{n-2}}{2!(1-x)^2}-$

$\frac{n(n-1)(n-2)}{3!(1-x)}.2^{n-3}+$ an integral expression of $(n-4)^{th}$ degree

coefficient of x^{n-1} in $(1-x)^{-4}$ is $\frac{n(n+1)(n+2)}{3!}$

" in $(1-x)^{-3}$ is $\frac{n(n+1)}{2!}$

" in $(1-x)^{-2}$ is n

" in $(1-x)^{-1}$ is 1

Hence the coefficient of x^{n-1} in $\frac{(1+x)^n}{(1-x)^4}$ is

$$\frac{2^n\, n(n+1)(n+2)}{3!} - \frac{2^{n-1}\, n^2(n+1)}{2!} + \frac{2^{n-2}\, n(n-1)}{2!}\cdot n - \frac{2^{n-3}\, n(n-1)(n-2)}{3!}$$

$$= \frac{2^{n-1}\, n(n+1)(n+2)}{3} - 2^{n-2}\, n^2(n+1) + 2^{n-3}\, n^2(n-1) - \frac{2^{n-4}\, n(n-1)(n-2)}{3}$$

$$= \frac{2^{n-4}}{3}\, n\, \{8(n+1)(n+2) - 12n(n+1) + 6n(n-1) - (n-1)(n-2)\}$$

$$= \frac{2^{n-4}}{3}\, n\, \{8n^2 + 24n + 16 - 12n^2 - 12n + 6n^2 - 6n - n^2 + 3n - 2\}$$

$$= \frac{2^{n-4}\, n}{3}\, \{n^2 + 9n + 14\}$$

$$= \frac{2^{n-4}\, n\, (n+2)(n+7)}{3}$$

$$= \frac{1}{3}\, n(n+2)(n+7)\, 2^{n-4}$$

Approximate values:

1. Find correct to six places of decimals the value of $\frac{1}{(9998)^{1/4}}$

Soln:

$$\left[\frac{1}{9998}\right]^{1/4} = \frac{1}{(10000-2)^{1/4}}$$

$$= \frac{1}{(10^4-2)^{1/4}}$$

$$= \frac{1}{10\left(1-\frac{2}{10^4}\right)^{1/4}}$$

$$= \frac{(1-2/10^4)^{-1/4}}{10}$$

$$= 1 + \frac{1}{4} \cdot \frac{2}{10^4} + \frac{\frac{1}{4} \cdot \frac{5}{4}}{2!} \cdot \frac{4}{10^8} + \cdots \Big/ 10$$

$$= \frac{1}{10} + \frac{1}{2} \cdot \frac{2}{10^5} + \frac{5}{4 \times 4 \times 2} \cdot \frac{4}{10^9} + \cdots$$

$$= \frac{1}{10} + \frac{1}{2} \cdot \frac{2}{10^5} + \frac{5}{8} \cdot \frac{1}{10^9} + \cdots$$

$$= 0.1 + \frac{1}{2}(0.00001) + \frac{5}{8}(0.000000001)$$

$$= 0.1 + 0.000005 + 0.0000000006$$

$$= 0.1000050006$$

$\therefore \frac{1}{(9998)^{1/4}} = 0.100005$ correct to six places of decimals.

2. Calculate correct to six places of decimals $(1.01)^{1/2} - (0.99)^{1/2}$

Soln:

Write $x = 0.01$

$$\therefore (1.01)^{1/2} = (1+x)^{1/2}$$

$$= 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2 + \frac{\frac{1}{2} \cdot -\frac{1}{2} \cdot -\frac{3}{2}}{3!}x^3 + \cdots$$

$$(0.99)^{1/2} = (1-0.01)^{1/2} \Rightarrow (1-x)^{1/2}$$

$$= 2\left\{\frac{1}{2}x + \frac{\frac{1}{2} \cdot -\frac{1}{2} \cdot -\frac{3}{2}}{3!}x^3 + \frac{\frac{1}{2} \cdot -\frac{1}{2} \cdot -\frac{3}{2} \cdot -\frac{5}{2} \cdot -\frac{7}{2}}{5!}x^5 + \cdots\right\}$$

$$= 2\left\{\frac{1}{2}x + \cancel{16}\,\frac{1}{16}x^3 + \frac{7}{256}x^5 + \cdots\right\}$$

$$= x + \frac{1}{8}x^3 + \frac{7}{128}x^5 + \cdots$$

$$= (0.01) + \frac{1}{8}(0.01)^3 + \frac{7}{128}(0.01)^5 + \cdots$$

$= 0.01 + \frac{1}{8}(0.000001) +$ terms not affecting the 8th decimal place

$$= 0.01 + 0.000000125$$

$$= 0.010000125$$

Logarithmic series.

1. Show that $\log \frac{a+x}{a-x} = \frac{2ax}{a^2+x^2} + \frac{1}{3}\cdot\left(\frac{2ax}{a^2+x^2}\right)^3 + \frac{1}{5}\left(\frac{2ax}{a^2+x^2}\right)^5 \cdots$

$$R.H.S = \frac{2ax}{a^2+x^2} + \frac{1}{3}\left(\frac{2ax}{a^2+x^2}\right)^2 + \frac{1}{5}\left(\frac{2ax}{a^2+x^2}\right)^5 + \cdots \quad (1)$$

We know that,

$$\frac{\log 1+x}{\log 1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$$

$\Rightarrow$ Multiply & divide by 2 in (1) we get,

$$= \frac{1}{2} \times 2\left[\frac{2ax}{a^2+x^2} + \frac{1}{3}\left(\frac{2ax}{a^2+x^2}\right)^3 + \frac{1}{5}\left(\frac{2ax}{a^2+x^2}\right)^5 + \cdots\right)$$

$$= \frac{1}{2}\log\left[\frac{1+\frac{2ax}{a^2+x^2}}{1-\frac{2ax}{a^2+x^2}}\right]$$

$= \frac{1}{2} \log \left[\frac{a^2+x^2+2ax}{a^2+x^2} \times \frac{a^2+x^2}{a^2+x^2-2ax} \right]$

$= \frac{1}{2} \log \frac{(a+x)^2}{(a-x)^2} \Rightarrow \frac{1}{2} \log \left[\frac{a+x}{a-x} \right]^2$

$= \log \left[\left(\frac{a+x}{a-x} \right)^2 \right]^{1/2} = \log \frac{a+x}{a-x} = L.H.S$

Hence proved.

2. P.T $\log \frac{n+1}{n-1} = \frac{2n}{n^2+1} + \frac{1}{3}\left[\frac{2n}{n^2+1}\right]^3 + \frac{1}{5}\left[\frac{2n}{n^2+1}\right]^5 + \dots$

R.H.S $= \frac{2n}{n^2+1} + \frac{1}{3}\left[\frac{2n}{n^2+1}\right]^3 + \dots$

We know that,

$\log \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)$

$\Rightarrow$ Multiply & divide by 2, we get

$= \frac{1}{2} \times 2 \left(\frac{2n}{n^2+1} + \frac{1}{3}\left(\frac{2n}{n^2+1}\right)^3 + \dots \right)$

$= \frac{1}{2} \log \left[\frac{1 + \frac{2n}{n^2+1}}{1 - \frac{2n}{n^2+1}} \right]$

$= \frac{1}{2} \log \left(\frac{n^2+1+2n}{n^2+1} \times \frac{n^2+1}{n^2+1-2n} \right)$

$= \frac{1}{2} \log \left[\frac{(n+1)^2}{(n-1)^2} \right]$

$= \log \left[\left(\frac{n+1}{n-1} \right)^2 \right]^{1/2}$

$= \log \frac{n+1}{n-1} = L.H.S.$

Hence proved.

3. S.T $\log \left(\frac{n+1}{n} \right)^{1/2} = \frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \dots$

R.H.S $= \frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \dots$

We know that,

$\log \left(\frac{1+x}{1-x} \right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)$

Multiply & divide by 2,

$= \frac{1}{2} \cdot 2\left(\frac{1}{2n+1} + \frac{1}{3} \cdot \frac{1}{(2n+1)^3} + \cdots\right)$

$= \frac{1}{2} \log\left(\frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}}\right)$

$= \frac{1}{2} \log\left[\frac{2n+1+1}{\cancel{2n+1}} \times \frac{\cancel{2n+1}}{2n+1-1}\right]$

$= \frac{1}{2} \log\left(\frac{2n+2}{2n}\right) = \frac{1}{2} \log\left[\frac{2(n+1)}{2n}\right]$

$= \frac{1}{2} \log\left(\frac{n+1}{n}\right)$

$= \log\left(\frac{n+1}{n}\right)^{1/2} = L.H.S$

Hence Proved

4. S.T $\log 3 = \log 2 + 2\left\{\frac{1}{5} + \frac{1}{3} \times \frac{1}{5^3} + \frac{1}{5} \cdot \frac{1}{5^5} + \cdots\right\}$

R.H.S $= \log 2 + 2\left\{\frac{1}{5} + \frac{1}{3} \times \frac{1}{5^3} + \cdots\right\}$

$= \log 2 + \log\left(\frac{1+1/5}{1-1/5}\right)$

$= \log 2 + \log\left(\frac{6}{\cancel{5}}^{3} \times \frac{\cancel{5}}{\cancel{4}_2}\right)$

$= \log 2 + \log\left(\frac{3}{2}\right)$

$= \log 2 + \log 3 - \log 2$

$= \log 3 = R.H.S$

Hence proved.

Signs of terms in the Binomial theorem:

Denoting the r^{th} term by U_r, we get

$\frac{U_{r+1}}{U_r} = \frac{n-r+1}{r} \cdot x = -\left[1 - \frac{n+1}{r}\right]x$.

If $x > 0$ and $r > n+1$, $\frac{U_{r+1}}{U_r}$ is positive

If $x < 0$ and $r > n+1$, $\frac{U_{r+1}}{U_r}$ is positive.

Numerically greatest term:

$$(1+x)^n = 1 + nx + \frac{n-1}{2!} \cdot x^2 + \dots + \frac{n(n-1)(n-2)\cdot(n-r+1)}{r!} x^r$$

$$U_{r+1} = \frac{n(n-1)(n-2)\dots(n-r+2)(n-r+1)}{r!} x^r$$

$$U_r = \frac{n(n-1)(n-2)\dots(n-r+2)}{(r-1)!} \cdot x^{r-1}$$

$$\frac{U_{r+1}}{U_r} = \frac{n-r+1}{r} \cdot x.$$

Example 1: Find the greatest term in the expansion of $(1+x)^{13/2}$ when $x = 2/3$.

$$U_{r+1} = \frac{\frac{13}{2}\left(\frac{13}{2}-1\right)\left(\frac{13}{2}-2\right)\dots\left(\frac{13}{2}-r+1\right)}{r!} \cdot x^r$$

$$U_r = \frac{\frac{13}{2}\left(\frac{13}{2}-1\right)\dots\left(\frac{13}{2}-r+2\right)}{(r-1)!} \cdot x^{r-1}$$

$$\frac{U_{r+1}}{U_r} = \frac{\frac{13}{2}-r+1}{r} \cdot x$$

$$= \frac{13-2r+2}{2r} \cdot x$$

$$= \frac{15-2r}{2r} \cdot x$$

$$= \frac{15-2r}{2r}\left[\frac{2}{3}\right] \qquad \left[\because \text{ when } x = \frac{2}{3}\right]$$

$$= \frac{15-2r}{3r}$$

$U_{r+1} \geq U_r$, if $\frac{15-2r}{3r} \geq 1$

(ie) $15-2r \geq 3r$

$15 \geq 5r$

$\frac{15}{5} \geq r$

$\boxed{r \leq 3}$