UNIT-5

5.6 Basis and Dimension:

Definition: A linearly independent subset S of a vector space V which spans the whole space V is called a basis of the vector space.

5.16-Theorem: Proof: Since V is finite dimensional there exists a

Pg. 5.19 finite subset S of V such that $L(S)=V$. By theorem 5.15 this set S contains a linearly independent subset $S'=\{v_1, v_2, \ldots, v_n\}$ such that

$$L(S')=L(S)=V$$

Hence S' is a basis for V.

5.17 Let V be a vector space over a field F. Then $S=\{$

Pg. 5.19 $v_1, v_2, \ldots v_n\}$ is basis for V iff every element of V can be uniquely expressed as linear combination of element of S.

Let S be basis for V.

Then by definition S is linearly independent and $L(S)=V$

Hence by theorem 5.13 every element of V can be uniquely expressed as a linear combination of element of S.

Conversely, suppose every element of V can be uniquely expressed as a linear combination of elements of S.

Clearly $L(S)=V$, Now, let $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$

Also, $0v_1 + 0v_2 + \cdots + 0v_n = 0$.

Thus we have expressed 0 as a linear combination of vectors of S in two ways.

$\therefore$ By hypothesis $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$

Hence S is linearly independent. Hence S is a basis

Theorem 5.18: Let V be a vector space over a field F. Let $S=$

Pg. 5.20 $\{v_1, v_2, \ldots v_n\}$ span V. Let $S=\{w_1, w_2, \ldots w_m\}$ be a linearly independent set of vectors in V. Then $m \leq n$.

Since $L(S) = V$, every vector in and in particular w_1 is a linear combination of $v_1 v_2 \ldots v_n$.

Hence $S_1 = \{w_1, v_1, v_2, \ldots v_n\}$ is a linear dependent set of vectors. Hence there exist a vector $v_k \neq w_1$ in S_1 which is a linear combination of the preceding vectors.

Let $S_2 = \{w_1, v_1 \ldots v_{k-1}, v_{k+1}, \ldots v_n\}$

Clearly $L(S_2) = V$. Hence w_2 is a linear combination of the vectors $\ldots v_{k-1}, v_{k+1}, \ldots v_n\}$ is linearly dependent in S_2. Hence $S_3 = \{w_2, w_1, v, \ldots v_{k-1}, v_{k+1}, \ldots v_n\}$ which is a linear combination. Hence there exists vector in S_3 which is a linear combination of the preceding vectors. since the w_i's are linearly independent this vector cannot be w_2 or w_1 and hence must be some v_j where $j \neq k$ (say, with $j > k$). Deletion of v_j from the set S_3 gives the set.

$S_4 = \{w_2, w_1, v_1, v_2, \ldots v_{k-1}, v_{k+1}, \ldots v_{j-1}, v_{j+1} \ldots v_n\}$ of n vectors spanning V. In this process at each step we insert one vector from $\{w_1, w_2, \ldots w_n\}$ and delete one vector from $\{v_1, v_2, \ldots v_n\}$. If $m > n$ after repeating this process n times, we arrive at the set $\{w_n, w_{n-1}, \ldots w_1\}$ which spans V. Hence $\{w_1, w_2, \ldots w_n, w_{n+1}, \ldots w_m\}$ is linearly dependent which is a contradiction. Here w_{n+1} is linear combination $w_1, w_2, \ldots w_n$.

Hence $m \leq n$.

Theorem 5.19 Any two bases of a finite dimensional vector space V have the same number of elements.

Since V is finite dimensional, it has a basis say $S = \{v_1, v_2, \ldots v_n\}$

Let $S' = \{w_1, w_2, \ldots w_m\}$ be any other basis for V.

Now $L(S) = V$ and S' is a set of m linearly independent vectors.

Hence by Theorem 5.18 $m \leq n$.

Also since $L(S') = V$ and S is a set of n linearly independent vector, $n \leq m$. Hence $m = n$.

Definition: Let V be finite dimensional vector space over a field F. The number of element in any basis of V is called the dimension of V and is denoted by dim V.

Theorem 5.20 Let V be vector space of dimension n. Then

(i) any set m vectors where $m>n$ is linearly dependent

(ii) any set m vector where $m<n$ cannot span V.

Proof: (i) Let $S=\{v_1, v_2 \dots v_n\}$ be a basis for V. Hence $L(S)=V$. Let S' be any set consisting of m vectors where $m>n$. Suppose S' is linearly independent since S spans V by theorem 5.8 $m \le n$ which is a contradiction.

(ii) Let S' be a set consisting of m vectors where $m<n$. Suppose $L(S')=V$. Now, $S=\{v_1, v_2, \dots v_n\}$ is a basis for V and hence linearly independent. Hence by theorem 5.18 $n \le m$ which a contradiction. Hence S' cannot span V.

Theorem 5.21 Let V be a finite dimensional vector space over a field F. Any linearly independent set of vector in V is part of a basis.

* Let $S=\{v_1, v_2, \dots v_r\}$ be a linearly independent set of vectors. If $L(S)=V$ then S itself is a basis. If $L(S) \ne V$ Choose an element $v_{r+1} \in V - L(S)$. Now, consider $S_1=\{v_1, v_2 \dots v_r, v_{r+1}\}$. we shall prove that S_1 is linearly independent by showing that no vector in S_1 is a linear combination of the preceding vectors (theorem 5.14)

* Since $\{v_1, v_2, \dots v_r\}$ is linearly independent v_i where $1 \le i \le r$ is not a linear combination of the preceding vectors.

Also $v_{r+1} \notin L(S)$ and hence v_{r+1} is not a linear independent.

Hence by Theo. 5.18 $\le m$ which is a contradiction Hence S' cannot span V.

Theorem 5.21 Let V be finite dimensional vector space over a field F. Any linearly independent set of vectors in V is part of a basis.

Let $S=\{v_1, v_2, \dots, v_r\}$ be a linearly independent set of vectors. If $L(S)=V$ then S itself is a basis.

If $L(S) \ne V$, choose an element $v_{r+1} \in V - L(S)$

Now, consider $S_1 = \{v_1, v_2, \ldots v_r, v_{r+1}\}$. We shall prove that S_1 is linearly independent by showing that no vectors in S_1 is a linear combination of the preceeding vectors. (5.14)

Since $\{v_1, v_2, \ldots v_r\}$ is linearly independent, v_i where $1 \le i \le r$, is not a linear combination of the preceeding vectors.

Also $v_{r+1} \notin L(S)$ and hence v_{r+1} is not a linear combination of $v_1, v_2, \ldots, v_r$. Hence S_1 is linearly independent.

If $L(S_1) = V$, then S_1 is a basis for V. If not we take an element $v_{r+2} \in V - L(S_1)$ and proceed as before. Since the dimension of V is finite, this process must stop at a certain stage giving the required basis containing S.

Theorem 5.22 Let V be a finite dimensional vector space over a field F. Let A be a subspace of V. Then there exists a subspace B of V such that $V = A \oplus B$.

Let $S = \{v_1, v_2, \ldots v_r\}$ be basis of A. By theorem 5.21, we can find $w_1, w_2, \ldots, w_s \in V$ such that $S' = \{v_1, v_2, v_3 \ldots v_r, w_1 \ldots\}$ is a basis of V.

Now let $B = L(\{w_1, w_2 \ldots w_s\})$

We claim that $A \cap B = \{0\}$ and $V = A + B$. Now, let $v \in A \cap B$. Then $v \in A$ and $v \in B$. Hence $v = \alpha_1 v_1 + \cdots \alpha_r v_r$.

$= \beta_1 w_1 + \cdots + \beta_s w_s$.

$\therefore \alpha_1 v_1 + \cdots + \alpha_r v_r - \beta_1 w_1 - \cdots - \beta_s w_s = 0$.

Now since S' is the identity independent $\alpha_i = 0 = \beta_j$ for all i and j. Hence $v = 0$ Thus $A \cap B = \{0\}$. Now let $v \in V$. Then $v = (\alpha_1 v_1 + \cdots \alpha_r v_r) + (\beta_1 w_1 + \cdots + \beta_s w_s) \in A + B$

Hence $A + B = V$ so that $V = A \oplus B$.

Definition: Let V be a vector space and $S = \{v_1, v_2 \ldots v_n\}$ be a set of independent vectors in V. Then S is called a maximal linearly independent set if for every $v \in V - S$, the set $\{v, v_1, v_2 \ldots v_n\}$ is linearly dependent.

Definition: Let $S = \{v_1, v_2, \ldots v_n\}$ be a set of vector in V and Let $L(S) = V$. Then S is called a minimual generating set if for any $v_i \in S$, $L(S - \{v_i\}) \neq V$.

Theorem 5.28 Let V be a vector space over a field F, let $S = \{v_1, v_2, \cdots v_n\} \subseteq V$. Then the following are equivalent

(i) S is a basis for V.

(ii) S is maximal linearly independent set.

(iii) S is a minimal generating set

Proof: (i) $\Rightarrow$ (ii) Let $S = \{v_1, v_2, \ldots v_n\}$ be a basis for V. Then by theorem 5.20 any $n+1$ vectors in V are linearly deperent and hence S is a maximal linearly independent set.

(ii) $\Rightarrow$ (i) Let $S = \{v_1 v_2 \cdots v_n\}$ be a maximal linearly independent set. Now to prove that S is a basis for V we shall show that $L(S) = V$.

obviously $L(S) \subset V$. Now let $v \in V$. If $v \in S$, then $v \in L(S)$. (since $S \subseteq L(S)$). If $v \notin S$, $S' = \{v_1, v_2, \ldots v_n, v\}$ is a linear independent set. $\therefore$ There exists a vector in S' which is a linear combination of preceding vectors. since $v_1, v_2, \ldots v_n$ are linealy independent, this vector must be v. Thus v is a linear combination of $v_1, v_2, \cdots v_n$. Therefore $v \in L(S)$. Hence $V \subseteq L(S)$. Thus $V = L(S)$.

(i) $\Rightarrow$ (iii) Let $S = \{v_1 v_2, \cdots v_n\}$ be a basis. Then $L(S) = V$. If S is not minimal, there exists $v_i \in S$ such that $L(S - \{v_i\}) = V$. Since S is linearly independent, $S - \{v_i\}$ is also linearly independent. Thus $S - \{v_i\}$ is a basis consisting of $n-1$ element which is contradiction. Hence S is a minimal generating set.

(iii) $\Rightarrow$ (i) Let $S = \{v_1, v_2, \cdots v_n\}$ be a minimal generating set. To prove that S is a basis, we have to show that S is linearly independent. If S is linearly dependent, there exists a vector v_k which is a linear combination of preceding vectors. Clearly $L(S - \{v_k\}) = V$ contradicting the minimality of S. Hence S is independent. $L(S) = V$, S is a basis for V

Theorem 5.24 Any vector space of dimension n over a field F is isomorphic to $V_n(F)$.

Proof: Let V be a vector space of dimension n. Let $\{v_1, v_2, \dots v_n\}$ be a basis for V. Then we known that if $v \in V$, v can be written uniquely as $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$ where $\alpha_i \in F$.

Now, consider the map $f : V \to V_n(F)$ given by $f(\alpha_1 v_1 + \dots + \alpha_n v_n) = (\alpha_1, \alpha_2, \dots \alpha_n)$. clearly f is 1-1 and onto

Let $v, w \in V$. Then $v = \alpha_1 v_1 + \dots + \alpha_n v_n$ and $w = \beta_1 v_1 + \dots + \beta_n v_n$.

$$f(v+w) = f[(\alpha_1+\beta_1)v_1 + \dots + (\alpha_n+\beta_n)v_n]$$
$$= [(\alpha_1+\beta_1), (\alpha_2+\beta_2), \dots, (\alpha_n+\beta_n)]$$
$$= (\alpha_1, \alpha_2, \dots \alpha_n) + (\beta_1, \beta_2 \dots \beta_n)$$
$$= f(v) + f(w)$$

Also $f(\alpha v) = f(\alpha\alpha_1 v_1 + \dots + \alpha\alpha_n v_n)$
$$= (\alpha\alpha_1, \alpha\alpha_2 \dots \alpha\alpha_n)$$
$$= \alpha(\alpha_1, \alpha_2, \dots \alpha_n)$$
$$= \alpha f(v)$$

Hence f is an isomorphism of V to $V_n(F)$.

Corollary: Any two vectors spaces of the same dimention over a field F are isomorphic, for if the vector spaces are of dimension n, each is isomorphism to $V_n(F)$ hence they are isomorphic.

Theorem 5.25 Let V and W be vector spaces over a field F. Let $T : V \to W$ be an isomorphism. Then T maps a basis of V onto a basis of W.

Let $\{v_1, v_2 \dots v_n\}$ be a basis for V. We shall prove that $T(v_1), T(v_2) \dots T(v_n)$ are linearly independent and they span W.

Now, $\alpha_1 (T(v_1)) + \alpha_2 T(v_2) + \dots + \alpha_n T(v_n) = 0$

$$\Rightarrow T(\alpha_1 v_1) + T(\alpha_2 v_2) + \dots + T(\alpha_n v_n) = 0$$
$$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n) = 0.$$
$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0 \quad (\text{since } T \text{ is 1-1})$$
$$\Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$$

(since $v_1, v_2, \dots v_n$ are linearly independent).

$\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

(Since $v_1, v_2, \dots v_n$ are linearly independent).

$\therefore T(v_1), T(v_2), \dots T(v_n)$ are linearly independent.

Now, let $w \in W$. Then since T is onto, there exists a vector $v \in V$ such that $T(v) = w$.

Let $v = \alpha_1 v_1 + \dots + \alpha_n v_n$

Then $w = T(v)$

$= T(\alpha_1 v_1 + \dots + \alpha_n v_n)$.

$= \alpha_1 T(v_1) + \dots + \alpha_n T(v_n)$.

Thus w is a linear combination of the vectors $T(v_1), \dots T(v_n)$.

$\therefore T(v_1), \dots T(v_n)$ span W and hence is a basis for W

Corollary: Two finite dimensional vector spaces V and W over a field F are isomorphic iff they have the same dimension.

Theorem: 5.26 Let V and W be finite dimensional vector spaces over a field F. Let $\{v_1, v_2, \dots v_n\}$ be a basis for V and let $w_1, w_2, \dots w_n$ be any n vectors in W (not necessarily) $T: V \to W$ i.e) $T(v_i) = w_i$, $i = 1, 2, \dots n$.

Let $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \in V$.

we define $T(v) = \alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_n w_n$

Now, let $x, y \in V$

Let $x = \alpha_1 v_1 + \dots + \alpha_n v_n$ and

$y = \beta_1 v_1 + \dots + \beta_n v_n$.

$\therefore x + y = (\alpha_1 + \beta_1) v_1 + \dots + (\alpha_n + \beta_n) v_n$

$\therefore T(x+y) = (\alpha_1 + \beta_1) w_1 + \dots + (\alpha_n + \beta_n) w_n$.

$= (\alpha_1 w_1 + \dots + \alpha_n w_n) + (\beta_1 w_1 + \dots + \beta_n w_n)$

$= T(x) + T(y)$.

Similarly $T(\alpha x) = \alpha T(x)$

Hence T is a linear transformation.

Also $v_1 = 1v_1 + 0v_2 + \dots + 0v_n$.

Hence $T(v_1) = 1w_1 + 0w_2 + 0w_n = w_1$

Similarly $T(v_i) = w_i$ for all $i = 1, 2, \ldots n$.

Now to prove the uniqueness let $T' : V \to W$ be any other linear transformation such that $T'(v_i) = w_i$.

Let $v = \alpha_1 v_1 + \ldots + \alpha_n v_n \in V$

$T'(v) = \alpha_1 T'(v_1) + \ldots \alpha_n (T')(v_n)$

$= \alpha_1 w_1 + \ldots + \alpha_n w_n = T(v)$.

Hence $T = T'$.

Remark: The above theorem shows that a linear transformation is completely determined by its values on the element of a basis.

Theorem 5.27: Let V be a finite dimensional vector space over a field F. Let W be a subspace of V. Then

(i) $\dim W \leq \dim V$; (ii) $\dim \frac{V}{W} = \dim V - \dim W$.

Proof: (i) Let $S = \{w_1, w_2, \ldots w_m\}$ be a basis for W. Since W is a subspace of V, S is part of a basis for V.

Hence $\dim W \leq \dim V$.

(ii) Let $\dim V = n$ and $\dim W = m$.

Let $S = \{w_1, w_2, \ldots w_n\}$ be a basis for W. Clearly S is a linearly independent set of vectors in V.

Hence S is a part of a basis in V. Let $\{w_1, w_2, w_n, v_1, v_2, \ldots v_r\}$ be a basis for V. Then $m + r = n$.

Now, we claim $S' = \{W + v_1, W + v_2, \ldots W + v_r\}$ is basis for $\frac{V}{W}$.

$\alpha_1 (W + v_1) + \alpha_2 (W + v_2) + \ldots + \alpha_r (W + v_r) = W + 0$

$\Rightarrow (W + \alpha_1 v_1) + (W + \alpha_2 v_2) + \ldots + (W + \alpha_r v_r) = W$

$\Rightarrow W + \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_r v_r = W$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_r v_r \in W$

Now since $\{w_1, w_2, \ldots w_n\}$ is a basis for W

$\alpha_1 v_1 + \ldots + \alpha_r v_r = \beta_1 w_1 + \ldots + \beta_n w_n$.

$\therefore \alpha_1 v_1 \ldots \alpha_r v_r - \beta_1 w_1 - \ldots - \beta_n w_m = 0$

$\therefore \alpha_1 = \alpha_2 = \alpha_r = \beta_1 = \beta_2 = \ldots \beta_m = 0$

$\therefore S'$ is a linearly independent set.

Now, let $W + v \in \frac{V}{W}$

Let $v = \alpha_1 v_1 + \ldots + \alpha_r v_r + \beta_1 w_1 + \ldots + \beta_m w_n$

Then $W+v = W + (\alpha_1 v_1 + \cdots + \alpha_r v_r + \cdots \beta_1 w_1 + \cdots + \beta_m w_m)$

$= W + (\alpha_1 v_1 + \cdots + \alpha_r v_r)$ (since $\beta_1 w_1 + \cdots + \beta_m w_m \in W$)

$= (W + \alpha_1 v_1) + \cdots + (W + \alpha_r v_r)$

$= \alpha_1 (W + v_1) + \cdots + \alpha_r (W + v_r)$.

Hence S' spans $\frac{V}{W}$ so that S' is a basis for $\frac{V}{W}$.

$\therefore \dim \frac{V}{W} = r = n - m$

$= \dim V - \dim W$.

Theorem 5.28 Let V be finite dimensional vector space over a field F. Let A and B be subspaces of V. Then $\dim(A+B) = \dim A + \dim B - \dim(A \cap B)$

Proof: A and B are subspaces of V. Hence $A \cap B$ is subspace of V. Let $\dim(A \cap B) = r$. Let $S = \{v_1, v_2, \ldots v_r\}$ be a basis for $A \cap B$. Since $A \cap B$ is a subspace of A and B, S is part of a basis for A and B. Let $\{v_1, v_2, \ldots v_r, u_1, u_2, \ldots u_s\}$ be a basis for A and $\{v_1, v_2, \ldots v_r, w_1, w_2, \ldots w_t\}$ be a basis for B.

We shall prove that $S' = \{v_1, \ldots v_r, u_1, \ldots u_s, \ldots w_1 \ldots w_t\}$ is a basis for $A+B$. Let $\alpha_1 v_1 + \cdots + \alpha_r v_r + \beta_1 u_1 + \cdots \beta_s u_s + \gamma_1 w_1 + \cdots \gamma_t w_t = 0$.

Then $\beta_1 u_1 + \cdots \beta_s u_s = -(\gamma_1 w_1 + \cdots + \gamma_t w_t) - (\alpha_1 v_1 + \cdots + \alpha_r v_r) \in B$

Hence $\beta_1 u_1 + \cdots + \beta_s u_s \in B$

Also $\beta_1 u_1 + \cdots + \beta_s u_s \in A$.

Hence $\beta_1 u_1 + \cdots + \beta_s u_s \in A \cap B$

$\therefore \beta_1 u_1 + \cdots + \beta_s u_s = \delta_1 v_1 + \cdots + \delta_r v_r$.

$\therefore \beta_1 u_1 + \cdots + \beta_s u_s - \delta_1 v_1 - \cdots - \delta_r v_r = 0$

$\therefore \beta_1 = \cdots = \beta_s = \delta_1 = \cdots \delta_r = 0$

(Since $\{u_1, u_s, v_1 \ldots v_r\}$ is linearly independent)

Similarly we can prove $\gamma_1 = \gamma_2 = \cdots \gamma_t = 0$.

$\therefore \alpha_i = \beta_j = \gamma_k = 0$ for $1 \le i \le r$, $1 \le j \le s$; $1 \le k \le t$.

Thus S' is a linearly independent set.

Clearly S' spans $A+B$.

$\therefore S'$ is a basis for $A+B$.

Hence $\dim(A+B) = r+s+t$.

Also $\dim A = r+s$; $\dim B = r+t$ and $\dim(A\cap B) = r$

$\dim A + \dim B - \dim A\cap B = (r+s) + (r+t) - r$

$= r+s+t$.

~~Corollary If V = A~~ $= \dim(A+B)$.

Corollary If $V = A\oplus B$, $\dim V = \dim A + \dim B$

$V = A\oplus B \Rightarrow A+B = V$ and $A\cap B = \{0\}$

$\therefore \dim(A\cap B) = 0$

Hence $\dim V = \dim A + \dim B$.

5.7. Rank and Nullity.

Definition: Let $T: V \to W$ be linear transformation. Then, the dimension of $T(V)$ is called the rank of T. The dimension of Ker T is called the nullity of T.

Theorem 5.29: Let $T: V \to W$ be a linear transformation. Then $\dim V =$ rank T + nullity T.

We know that $V/\ker T = T(V)$

$\therefore \dim V - \dim(\ker T) = \dim(T(V))$

$\therefore \dim V -$ nullity T = rank T.

$\therefore \dim V =$ nullity T + rank T.

Note: Ker T is also called null space of T

Definition: A linear transformation $T: V \to W$ is called non-singular if T is 1-1; otherwise T is called singular.

5.8 Matrix of a linear Transformation:

(Solved problem in 5 22 pg.no) - 8 sums

Definition: Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $m\times n$ matrices. We define the sum of these two matrices by $A+B = (a_{ij}+b_{ij})$

Note that we have defined addition only for two matrices having the same number of rows and the same number of columns.