UNIT-II

*: (1) Theorem: 5M 2011.

Let f be defined on $[a,b]$ if f is differentiable at a point $x \in [a,b]$ then f is continuous at x.

Proof:

Given that f is differentiable at a point $x \in [a,b]$

$\lim_{t \to x} \frac{f(t) - f(x)}{t - x}$ exists and is equal to $f'(x)$

Now if $t \neq x$ then we may write

$$f(t) - f(x) = \frac{f(t) - f(x)}{t - x}(t - x).$$

Taking limit $t \to x$ on both sides in the above equation

we get,

we are going to prove $\lim_{t \to x} f(t) = f(x)$.

$$\lim_{t \to x} |f(t) - f(x)| = \lim_{t \to x} \left[\frac{f(t) - f(x)}{t - x} (t - x) \right]$$

$$= \lim_{t \to x} \left[\frac{f(t) - f(x)}{t - x} \right] \lim_{t \to x} (t - x)$$

$$= f'(x) \cdot 0$$

$$= 0$$

$$\lim_{t \to x} f(t) - f(x) = 0$$

$$\lim_{t \to x} f(t) = f(x).$$

Theorem : 5.3

Suppose f & g are deffined on $[a,b]$ and are differentiable at a point $x \in [a,b]$ then $f+g$, fg & f/g are differentiable at x and

a) $(f+g)'(x) = f'(x) + g'(x)$.

b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

c) $$(f/g)'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}.$$

in (c) we assume of course that $g(x) \neq 0$.

Proof :

Let us prove,

i) $(f+g)'(x) = f'(x) + g'(x)$.

Give that f and g are differentiable on $[a,b]$,

$$\lim_{t\to x} \frac{f(t)-f(x)}{t-x} = f'(x) \ \& \ \lim_{t\to x} \frac{g(t)-g(x)}{t-x} = g'(x).$$

$$\lim_{t\to x} \left[\frac{f(t)-f(x)+g(t)-g(x)}{(t-x)}\right] = f'(x)+g'(x).$$

$$\lim_{t\to x} \left[\frac{f(t)+g(t)-[f(x)+g(x)]}{t-x}\right] = f'(x)+g'(x).$$

$$\lim_{t\to x} \frac{(f+g)(t) - (f+g)(x)}{t-x} = f'(x) + g'(x).$$

$$(f+g)'(x) = f'(x) + g'(x).$$

Hence (i) the proof.

consider

$$h(t)-h(x) = \frac{f(t)}{g(t)} - \frac{f(x)}{g(x)}$$

$$= \frac{1}{g(t).g(x)}\left[f(t)g(x) - f(x)g(x) + f(x)g(x) - g(t)f(x)\right]$$

$$= \frac{1}{g(t)g(x)}\left[g(x)\left[f(t)-f(x)\right] + f(x)\left[g(x)-g(t)\right]\right]$$

$$\lim \frac{h(t)-h(x)}{t-x} \times \cancel{t-x} = \frac{1}{g(t).g(x)}\left\{\left[\lim_{t\to x} g(x)\left[\frac{f(t)-f(x)}{t-x}\right]\right.\right.$$

$$\left.\left. - \lim_{t\to x} f(x)\left[\frac{g(t)-g(x)}{t-x}\right]\right]\right\}$$

$h'(x)$.

$$h'(x) = \lim_{t\to x} \frac{1}{g(t).g(x)} \cdot g(x).f'(t) - f(x)g'(t)\}$$

$$h'(x) = \frac{g(x)f'(t) - f(x)g'(t)}{[g'(x)]^2}.$$

Theorem: 5.5 (chain rule)

10M

suppose f is continuous on $[a,b]$. $f'(x)$ exist at some point $x \in [a,b]$, g is defined on an interval I which contains the range of f and g is differentiable at the point $f(x)$. if

$$h(t) = g(f(t)) \quad (a \leq t \leq b).$$

then h is differentiable at x and

$$h'(x) = g'(f(x))\, f'(x).$$

proof:

f is continuous on $[a,b]$ and $f'(x)$ exists at some point $x \in [a,b]$

since g is defined on I. diff at $f(x)$ &

$h(t) = g[f(t)]$.

Now we have to prove that h is differentiable at x.

Let $\boxed{y = f(x)}$ given that f is diff at x, By deffinition of the derivative.

$$\frac{f(t) - f(x)}{t-x} = f'(x) + u(t).$$

$$\lim_{t \to x} f(t) - f(x) = (t-x)[f'(x) + u(t)]$$

where $u(t) \to 0$ as $t - x \to$ ①

where $t \in [a,b]$

since g is diff at $f(x)$,

where $v(s) \to 0$, $s \to y$ and $s \in I$

Let us $f(t)$ using operation ① & ② we get,

$$h(t) - h(x) = [g\, f(t) - g(f(x))]$$

where $v(s) \to 0$ as $s \to y$ and $s \in I$ Let $\boxed{s = f(t)}$.

using eqn ① & ② we get,

$$h(t) - h(x). = g[f(t) - g(f(x))]$$

$= g(s) - g(y)$

$= s-y \, [g'(y) + v(s)]$

$= [f(t) - f(x)] \, [g'(y) + v(s)]$

$= (t-x) \, [f'(x) + u(t)] \, [g'(y) + v(s)]$

if $t \neq x$.

$\div$ by $(t-x)$.

$$\frac{h(t)-h(x)}{t-x} = [f'(x) + u(t)] \, [g'(y) + v(s)]$$

$$\frac{h(t)-h(x)}{t-x} = f'(x)g'(y) + f'(x)v(s) + u(t)g'(y) + u(t)v(s).$$

Taking $\lim_{t \to x}$ and $s \to y$ in the above equation

we get $h'(x) = g'(y) f'(x)$.

$h'(x) = g'[f(x)] f'(x)$

Hence the theorem.

2011. 5M(1).

Theorem: 5.8 (MEAN VALUE THEOREM)

Let f be defined on $[a,b]$ if f has a local maximum at a point $x \in [a,b]$ and if $f'(x)$ exists then $f'(x) = 0$.

Proof:

Let $x \in (a,b)$

choose by the definition of local maximum

$a < x-\delta < x < x+\delta < b$ if $x-\delta < t < x$ then

f is local maximum of x.

$t \le x \Rightarrow t - x \le 0$ (-ve)

$\Rightarrow f(t) \le f(x) \Rightarrow f(t) - f(x) \le 0$

$\therefore \dfrac{f(t) - f(x)}{t - x} \ge 0 \longrightarrow ①$

Taking limit $t \to x$ in the above eqn, we get $f'(x) \ge 0$

if $x < t < x + \delta$ then $\dfrac{f(t) - f(x)}{t - x} \le 0$,

Taking limit as $t \to x$.

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x} \le 0,$$

$$f'(x) \le 0 \longrightarrow ②.$$

① & ②

if f has a local maximum at a point $x \in [a,b]$

then $f'(x) = 0$.

Hence the theorem.

Theorem: 5.9 (Generalized mean value theorem)

If f and g are continuous real function on $[a,b]$ which are differentiable in (a,b) then there is a point $x \in (a,b)$, at which

$$[f(b) - f(a)]\, g'(x) = [g(b) - g(a)]\, f'(x).$$

Note that differentiability is not required at the endpoints.

proof:

Given that f & g are continuous real valued function on [a,b] which are diff in (a,b)

Let $h(t) = [f(b)-f(a)]g(t) - [g(b)-g(a)]f(b) \rightarrow (1)$

$a \leq t \leq b$.

Then h is continuous on [a,b] and h is diff in (a,b)

Next to P.T

$h(a) = h(b)$

In equation (1) put $x=a$, $t=a$ we get,

$h(a) = [f(b)-f(a)]g(a) - [g(b)-g(a)]f(a)$

$= f(b)g(a) - \cancel{f(a)g(a)} - g(b)f(a) + \cancel{g(a)f(a)}$

$= f(b)g(a) - g(b)f(a) \rightarrow (3)$

in eqn (1) put $t=b$, $x=b$ we get,

$h(b) = [f(b)-f(a)]g(b) - [g(b)-g(a)]f(b)$

$= \cancel{f(b)g(b)} - f(a)g(b) - \cancel{g(b)f(b)} + g(a)f(b)$

$= f(a)g(b) - g(a)f(b) \rightarrow (4)$

here (3) & (3) are equal

$\therefore h(a) = h(b)$.

To complete the theorem next we have to prove that $h'(x) = 0$ for some $x \in (a,b)$

if h is constant this hold for every $x \in (a,b)$

case (i):

if $h(t) > h(a)$ for some $t \in (a,b)$. Let be a point on $[a,b]$ at which h attains its maximum value for (known theorem)

If t has a local maximum at a point $x \in (a,b)$ and if $f'(x)$ exists then $f'(x)$

$\Rightarrow h'(x) = 0$.

case (ii):

if $h(t) < h(a)$ for some point $t \in (a,b)$ the same argument applies it. we choose for a point on $[a,b]$ where h attains its maximum value

$\therefore h'(x) = 0$

eqn ① becomes.

$\boxed{[f(b) - f(a)]} g'(x) - [g(b) - g(a)] f'(x) = 0$

$[f(b) - f(a)] g'(x) = [g(b) - g(a)] f'(x)$.

The above theorem is known as a generalised mean value theorem.

Definition:

The intersection of the set E_α is defined to be the set P such that $x \in P$ iff $x \in E_\alpha$ for all α in A. we used the notation.

$P = \bigcap_{\alpha \in A} E_\alpha$ (or)

$P = \bigcap_{m=1}^{n} E_m$ (or) $P = E_1 \cap E_2 \cap \cdots E_n$ (or)

$P = \bigcap_{m=1}^{\infty} E_m$.

Theorem : 5.10.

[Lagrange's mean value theorem]

Statement:

If f is real continuous function on $[a,b]$ which is differentiable in (a,b) then there is a point $x \in (a,b)$ at which,

$$f(b) - f(a) = (b-a) f'(x).$$

Proof:

Given that f is continuous function on $[a,b]$ and diff in (a,b) we have to P.T by known result.

If f and g are continuous real function on $[a,b]$ which are diff in (a,b) then there is a point $x \in (a,b)$ at which.

$$(f(b) - f(a)) g'(x) = (g(b) - g(a)) f'(x) \longrightarrow (1)$$

let $g(x) = x$. & $g'(x) = 1$

$g(a) = a$, $g(b) = b$

Put $g(x) = x$ in (1)

$g'(x) = 1$, $g(a) = a$, $g(b) = b$ in we get (1)

$$(1) \Rightarrow f(b) - f(a) = (b-a) f'(x)$$

Hence the proof.

Theorem: 5.11

Suppose f is differentiable in (a,b)

a) If $f'(x) \geq 0$ for all $x \in (a,b)$ then f is monotonically increasing.

b) If $f'(x) = 0$ for all $x \in (a,b)$ then f is constant.

c) If $f'(x) \leq 0$ for all $x \in (a,b)$ then f is monotonically decreasing.

Proof:

Given that f is differentiable in (a,b) then such that a point $x \in (a,b)$

$\Rightarrow f(b) - f(a) = f'(x)(b-a) \longrightarrow (A)$.

[$\because$ lagrange's theorem

Here $x_1, x_2 \in (a,b)$ for some x between x_1 & x_2

Let $f(x_2) - f(x_1) = (x_2 - x_1) f'(x) \longrightarrow (1)$

$a = x_1$, $b = x_2$ in (A)

Sequence increases for increase monotonically increasing $x_1 < x_2 < x_3 \ldots$

Let $f'(x) \geq 0$ then

$(1) \Rightarrow f(x_2) - f(x_1) \geq 0$

$f(x_2) \geq f(x_1)$

$\Rightarrow f$ is monotonically increasing.

ii) Let $f'(x) = 0$

Then (1) $f(x_2) - f(x_1) = 0$

$f(x_2) = f(x_1)$

f is constant.

iii) Let $f'(x) \leq 0$

Then ① $\Rightarrow f(x_2) - f(x_1) \leq 0$

$$f(x_2) \leq f(x_1)$$

f is monotonically decreasing.

Theorem: 5.12

Intermediate value theorem.

Statement:

Suppose f is a real differentiable function on $[a,b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a,b)$ such that $f'(x) = \lambda$.

A similar result holds of course if, $f'(a) > f'(b)$.

Proof:

Given that f is a diff function on $[a,b]$

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x} = f'(x).$$

Put $g(t) = f(t) - \lambda(t) \longrightarrow ①$

$g'(t) = f'(t) - \lambda$

$g'(a) = f'(a) - \lambda$

Since $f'(a) < \lambda < f'(b)$.

$f'(a) < \lambda$

$f'(a) - \lambda < 0$

$g'(a) < 0$.

[by previous theorem g is M.D] so that

$g(t_1) < g(a)$ for some $t_1 \in (a,b)$ let ~~g(b)~~

let $g'(b) = f'(b) - \lambda$

since, $\lambda < f'(b)$.

$\Rightarrow 0 < f'(b) - \lambda$

$\Rightarrow f'(b) - \lambda > 0$

$\Rightarrow g'(b) > 0$

[previous theorem by part g is M.I] so that

$g(t_2) < g(b)$ for some $t_2 \in (a,b)$ hence g

attains its minimum value on $[a,b]$. w.k.t

let f be defined on $[a,b]$ if f has a

local maximum at a point $x \in (a,b)$ and if

$f'(x)$ exists then $f'(x) = 0$.

$g'(x)$ also exists for some point x such

that. $a < x < b$.

$\Rightarrow g(x) = 0$

$g(x) = f(x) - \lambda(x)$

$g'(x) = f'(x) - \lambda$

$f'(x) = \lambda$

$x \in (a,b)$

$f'(x) = \lambda$.

Hence the proof.

2011 Nov * Theorem: 5.15.

Taylor's theorem.

Statement:

Suppose f is a real function on $[a,b]$ n is a positive integer, $f^{(n-1)}$ is continuous on $[a,b]$ $f^n t$ exists for every $t \in (a,b)$. Let α, β be defined distinct points $[a,b]$ and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (t-\alpha)^k$$

Then there exist a point x between α and β such that,

$$f(\beta) = P(\beta) + \frac{f^n(x)}{n!} (\beta-\alpha)^n.$$

For $n=1$ this is just the mean value theorem in general the theorem shows that f can be approximated by a polynomial of degree $n-1$ and that allows us to estimate the error. If we know bounds on $|f^n(x)|$

Proof:

Let M be the number defined by

$$f(\beta) = P(\beta) + M(\beta-\alpha)^n \rightarrow (3).$$

Let the $g(t) = f(t) - P(t) - M(t-\alpha)^n \rightarrow (4).$

Where $a \leq t \leq b$.

Now we have to P.T.

$n! M = f^n(x)$.

for some x between α & β.

Now to find the n^{th} derivative of the eqn ③ we get,

$$g'(t) = f'(t) - p'(t) - nM(t-\alpha)^{n-1}$$

$$g''(t) = f''(t) - p''(t) - n(n-1)M(t-\alpha)^{n-2}$$

$$g'''(t) = f'''(t) - p'''(t) - n(n-1)(n-2)M(t-\alpha)^{n-3}$$

In general.

$$g^n(t) = f^n(t) - p^n(t) - n(n-1)(n-2)M(t-\alpha)^{n-n}$$

$$= f^n(t) - p^n(t) - n!\,M$$

$$= f^n(t) - 0 - n!\,M$$

$$g^n(t) = f^n(t) - n!\,M.$$

$$p(t) = \sum_{k=0}^{n-1} \frac{f^k(\alpha)}{k!}(t-\alpha)^k.$$

$$p(t) = \frac{f^0(\alpha)(t-\alpha)}{0!} + \frac{f'(\alpha)}{1!}(t-\alpha) + \frac{f''(\alpha)}{2!}(t-\alpha)^2 + \frac{f^{n-1}(\alpha)}{(n-1)!}(t-\alpha)^{n-1}$$

$$p(t) = 1 + \frac{f'(\alpha)}{1!}(t-\alpha) + \cdots + \frac{f^{n-1}(\alpha)}{(n-1)!}(t-\alpha)^{n-1}$$

$$p'(t) = 0 + f'(\alpha) + \frac{f^2(\alpha)}{2!}2(t-\alpha) + \cdots + \frac{f^{n-1}(t-\alpha)^{n-2}}{(n-1)!}(n-1)$$

$p'(t) = f'(\alpha) + f^2(\alpha)(t-\alpha) + \cdots + \frac{f^{n-1}(\alpha)}{(n-1)!\,(n-2)!}$

$(n-1)(n-2)(t-\alpha$

$P^n(n) = 0$

$(t-\alpha)^n = n(t-\alpha)^{n-1}$

$= n(n-1)(t-\alpha)^{n-2} \cdots n(n-1)(n-2)\cdots 2.1 = n!$

$(t-\alpha)^n = n!$

Next to complete the theorem-11 we can show that $g^n(x) = 0$ for some x between α and β

Since $p^k(\alpha) = f^k(\alpha)$ for $k = 0, 1, \ldots n-1$

i.e. $p(\alpha) = f(\alpha)$, $p'(\alpha) = f'(\alpha)$. $p''(\alpha) = f''(\alpha)$

$t = \alpha$ in eqn (4).

$g(\alpha) = f(\alpha) - p(\alpha)$

$g(\alpha) = g'(\alpha) \cdots g^{n-1}(\alpha) = 0$

put $t = \beta$ in (4) we get,

$g(\beta) = p(\beta) + M(\beta-\alpha)^n - p(\beta) - m(\beta-\alpha)^n$

$g(\beta) = 0.$

Since $g'(x_1) = 0$ we conclude that $g''(x_2) = 0$ for some x_2 between α & x_1. After n steps we arrive at the conclusion that $g^n(x_n) = 0$.

our choice of M shows that $g(\beta) = 0$ so that $g'(x_1) = 0$ for some between α & β (by mean value theorem)

If f is continuous on $[a,b]$ & f is ~~derivable~~ derivable on $[a,b]$ & $f(a)=f(b)$ then $\exists$ real no. between a & b $f'(c)=0$

put ③ in ⑤ we get,

between α & β

$f_1^n(x)=0$

$f^n(x)-n!M=0$

$$M=\frac{f^n(x)}{n!}$$

$$f(\beta)=p(\beta)+\frac{f^n(x)}{n!}(\beta-\alpha)^n$$

Hence theorem.

Theorem : 5.19

suppose f is continuous mapping of $[a,b]$ into R^k and f ~~does~~ is differentiable on (a,b) then there exists $x\in(a,b)$ such that,

$$|f(b)-f(a)|\le(b-a)|f'(x)|$$

Proof:

Given that f is differentiable in (a,b)

Put $z=f(b)-f(a)\longrightarrow$ ①

define $\phi(t)=zf(t)\longrightarrow$ ②.

where $a\le t\le b$.

hence ϕ is a real valued continuous in (a,b)

... mean value theorem

Let f be continuous function on $[a,b]$ & diff in (a,b) $f(a)=f(b)$ then there exists a point $c \in (a,b)$ such that,

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

There exists a point $x \in (a,b)$ such that

$$\frac{\phi(b)-\phi(a)}{b-a} = \phi'(x) \longrightarrow ③$$

in ② put $t=a$ and b we get.

$$\phi(t) = z \cdot f'(t).$$

$$\phi(b)-\phi(a) = (b-a) z f'(t)$$

$$= (b-a) z \frac{f(b)-f(a)}{b-a}$$

$$\phi(b)-\phi(a) = (b-a) z$$

$$\phi(b) = z f(b).$$

$$\phi(a) = z f(a)$$

substracting the above two eqn we get,

$$\phi(b)-\phi(a) = z f(b) - z f(a)$$

$$= z \cdot \underline{f(b)-f(a)}_{z}$$

$$= z \cdot z$$

$$\phi(b)-\phi(a) = |z^2| \longrightarrow ④.$$

Now diff ② with respect to t we get

$$\phi'(t) = z f'(t) \longrightarrow ⑤.$$

$$③ \Rightarrow (\phi(b)-\phi(a) = (b-a)\phi'(z).$$

$$③ \Rightarrow (\phi(b)-\phi(a) = (b-a) z f'(x). \rightarrow ⑥$$

Now ④ in above eqn, we get.

$$|z^2| = (b-a) \cdot z f'(x).$$

Applying such that ⑤ in equality we get

Sub ⑥ in ④

$$|z^2| \le (b-a)|z|\,|f'(x)|$$ (schwarz's inequality)

$$|z| \le (b-a)\,|f'(x)|$$

$$\Rightarrow |f(b) - f(a)| \le (b-a)\,|f'(x)|$$

L' HOSPITAL'S RULE

Statement:

Suppose f and g are real and differentiable in (a,b) and $g'(x) \neq 0$ for all $x \in (a,b)$ where $-\infty \le a < b \le +\infty$ suppose,

$$\frac{f'(x)}{g'(x)} \to A \text{ as } x \to a$$

if $f(x) \to 0$ and $g(x) \to 0$ as $x \to \infty$

or if $g(x) \to +\infty$ as $x \to a$

then, $\dfrac{f(x)}{g(x)} \to A$ as $x \to a$

Proof:

Now first consider the case in which

$-\infty \le A \le +\infty$

Choose a real numbers of such that $A < q$

and choose r such that $A < r < q$

By theorem:

There is a point $c \in (a,b)$ such that $a < x < x$.

$$\Rightarrow \frac{f'(x)}{g'(x)} < r \rightarrow (5)$$

If $0 < m < y < c$ then by g.m.v theorem

$[f(b) - f(a)] g'(x) = [g(b) - g(a)] f'(x)$.

show that there is a point $t \in (x, y)$ such

that,

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x)}{g'(x)} \Rightarrow \frac{-[f(a) - f(b)]}{-[g(a) - g(b)]} = \frac{f'(x)}{g'(x)}$$

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)} < r \rightarrow (6)$$

we get,

$$\frac{f'(y)}{g'(y)} < r < q \quad (a < y < c) \rightarrow (7).$$

next suppose (2) holds keeping y fixed

in (4) we can choose a point $c \in (a,b)$ such that

$g(x) > g(y)$ and $g(x) > 0$ if $a < x < c$

Multiplying by $\frac{(g(x)-g(y))}{g(x)}$ we get,

$$\frac{f(x)-f(y)}{g(x)-g(y)} \cdot \frac{g(x)-g(y)}{g(x)} < r\left\{\frac{g(x)-g(y)}{g(x)}\right\}$$

$$\frac{f(x)}{g(x)} - \frac{f(y)}{g(x)} < r\left\{1-\frac{g(y)}{g(x)}\right\}$$

$$\frac{f(x)}{g(x)} < r - r\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)} \quad (a<x<c_1) \to (8)$$

Let $x \to a$ in (8) $g(x) \to \infty$ as $x \to a$ show that there is a point (a,c) such that,

$$\frac{f(x)}{g(x)} < q \quad (a<x<c_2) \to (9)$$

Summation a (7) & (8) such that for any q subject only to the condition $A<q$ there is a points c_2 such that $\frac{f(x)}{g(x)}<q$ if $a<x<c_2$

In the same way if $-\infty < A < +\infty$ and p is choosen so that $p<A$ we can find a point c_3 such that,

$$p < \frac{f(x)}{g(x)} \quad (a<x<c_3) \to (6)$$

From ⑨ & ⑩ we get.

$p < \frac{f(x)}{g(x)} < q$

$\frac{f(x)}{g(x)}$ lies between p & q already we know that

A lies between p & q also

$\frac{p(x)}{g(x)} \to A$ as $x \to a$.

UNIT – III

Recall Rieman of integral.

Definition :

Let [a,b] be the given integral by a partition P of [a,b] we mean a finite set of points.

$(x_0, x_1, \dots x_n)$

where $a = x_0 \le x_1 \le \dots \le x_n = b$

Let $[x_{r-1}, x_r]$ be the r^{th} sub integral where

$\Delta x_r = x_r - x_{r-1}$

$=$ length of Δx_r

Let $M_r = \sup m\ f(x)$ in Δx_r

$m_r = \text{Inf}\ m\ f(x)$ in Δx_r

consider the sum $= \sum_{r=1}^{n} M_r \Delta x_r$

= upper sum of corresponding to the partition n

$U(P, f, \alpha)$

consider the sum $= \sum_{r=1}^{m} m_r \Delta x_r$

= lower sum of the correspond-ing to the partition.