$\le \lim_{n \to \infty} U(P_n, f_2, \alpha)$

$\le \int_a^b f_2 \, d\alpha.$

$$\int_a^b f_1 \, d\alpha = \int_a^b f_2 \, d\alpha.$$

c) S.T $f \in R(\alpha)$ on $[a,b]$ and $a < c < b$ P.T $f \in R(\alpha)$ on $[a,c]$ and $f \in R(\alpha)$ on $[c,d]$ and $\int_a^b f \, d\alpha = \int_a^c f \, d\alpha + \int_c^b f \, d\alpha.$

proof:

Let P be a partition of $[a,b]$

c is any point b/w a & b

let $P_1 = P \cap [a,c]$

$P_2 = P \cap [c,d]$

By data $f \in R(\alpha)$ on $[a,b]$

$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$

or $U(P_1,f,\alpha) + U(P_2,f,\alpha) - [L(P_1,f,\alpha) + L(P_2,f,\alpha)] < \epsilon$

for $(P = P_1 + P_2)$

or) $[U(P_1,f,\alpha) - L(P_1,f,\alpha) + U(P_2,f,\alpha) - L(P_2,f,\alpha)]$

$\Rightarrow U(P_1,f,\alpha) - L(P_1,f,\alpha) < \epsilon$ and

$U(P_2,f,\alpha) - L(P_2,f,\alpha) < \epsilon$

$f \in R(\alpha)$ on $[a,c]$ & $f \in R(\alpha)$ on $[c,b]$ To P.T

$$\int_a^b f \, d\alpha = \int_a^c f \, d\alpha + \int_c^b f \, d\alpha.$$

we have $U(P,f,\alpha) = U(P_1,f,\alpha) + U(P_2 \cdot f, \alpha)$

Take the infimum on both sides as partitions become finer.

$$\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha.$$

Hence the proof.

d) S.T. $\left|\int_a^b f\,d\alpha\right| \le M\{(\alpha(b)-\alpha(a)\}$

Proof:

consider $\int_a^b f\,d\alpha$ = Inf of $U(P, f, \alpha)$ over P.

Since $f \in R(\alpha)$,

$= \underset{\text{over } P}{\text{Inf}} \cdot \lim_{n\to\infty} \sum_{i=1}^{n} M_i \Delta\alpha_i$ where M_i is supremum of $f(x)$ in $\Delta\alpha_i$ (or)

$$\left|\int_a^b f\,d\alpha\right| \le \lim_{n\to\infty} \sum_{i=1}^{n} |M_i| \Delta\alpha_i.$$

For mod of sum $\le$ sum of modulus.

$$\le M \lim_{n\to\infty} \sum_{i=1}^{n} \Delta\alpha_i \text{ for } |f(a)| \le M \text{ on } [a,b]$$

$$\le M[\alpha(b) - \alpha(a)]$$

e) $\int_a^b f\,d(\alpha_1+\alpha_2) = \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2$

Proof:

Since $f \in R(\alpha)$,

$$\int_a^b f\,d\alpha_1 = \underset{\text{over } P}{\text{Inf}}\ U(P, f, \alpha)$$

$$= \underset{\text{over } P}{\text{Inf}}\ \lim_{n\to\infty} \sum_{i=1}^{n} M_i\,\Delta\alpha_i \rightarrow (1)$$

where M_i = sup of $f(x)$ in $\Delta\alpha_i$.

Since $f \in R(\alpha_2)$

$$\int_a^b f\,d\alpha_2 = \underset{\text{over } P}{\text{Inf}}\ U(P, f, \alpha_2)$$

$$= \underset{\text{over } P}{\text{Inf}}\ \lim_{n\to\infty} \sum_{i=1}^{n} M_i\,\Delta\alpha_{2i} \rightarrow (2)$$

From (1) & (2) $\Rightarrow$

$$\int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2 = \underset{\text{over } P}{\text{Inf}}\ \lim_{n\to\infty} \left[\sum_{i=1}^{n} M_i\,\Delta\alpha_i + M_i\,\Delta\alpha_{2i}\right]$$

exists

$$= \underset{\text{over } P}{\text{Inf } M}\ \lim_{n\to\infty} \sum_{i=1}^{n} M_i\,[\Delta\alpha_{1i} + \Delta\alpha_{2i}] \text{ exists}$$

$$= \underset{\text{over } P}{\text{Inf } M}\ U(P, f, (\alpha_1+\alpha_2)) \text{ exists}$$

$$= \int_a^b f\,d(\alpha_1+\alpha_2)$$

$= f \in R(\alpha)$.

f) S.T $\int_a^b f\,d(c\alpha) = c\int_a^b f\,d\alpha$.

Proof:

Since $f \in R(\alpha)$ on $[a, b]$

$$\int_a^b f\,d\alpha = \underset{\text{over } P.}{\text{Inf}}\ U(P, f, \alpha) \text{ exists} \rightarrow (1)$$

consider

$$\int_a^b f\,d\alpha = \inf_{\text{over } P} U(P, f, c\alpha)$$

$$= \inf_{\text{over } P} \lim_{n\to\infty} \sum_{i=1}^{n} M_i \Delta c\alpha_i$$

$$= \inf_{\text{over } P} c \lim_{n\to\infty} \sum_{i=1}^{n} M_i \Delta\alpha_i \text{ exists by } ①$$

$\int_a^b f\,d\alpha$ exists and $\int_a^b f\,d(c\alpha)$

$$= c\int_a^b f\,d\alpha.$$

Theorem: 6.13.

a) $fg \in R(\alpha)$

b) $\left|\int_a^b f\,d\alpha\right| \le \int_a^b |f|\,d\alpha.$

Proof:

Let $\phi(t) = t^2$ $\quad \phi(t) = t^2$

$\phi[t(x)] = [f(x)]^2.$ $\quad \phi(f(x)) = (f(x))^2$

Since $f(x) \in R(\alpha)$

$\phi(x) = [f(x)]^2 \in R(\alpha)$ by ① theorem

we have,

$(f+g)^2 - (f-g)^2 = 4fg$

Since f and $g \in R(\alpha)$

$f+g$ and $f-g \in R(\alpha)$.

$(f+g)^2$ & $(f-g)^2 \in R(\alpha)$ by ①

L.H.S $\in R(\alpha)$

R.H.S $\in R(\alpha)$

i.e: $4fg \in R(\alpha)$ (or) $fg \in R(\alpha)$

b) Let $\phi(t) = |f|$

$\Rightarrow \phi\, |f(x)| = |f(x)|$ $\phi(t) = |f|$

$\phi|f(x)| = |f(x)|$

since $f(x) \in R(\alpha)$

$\phi\,[f(x)] \in R(\alpha)$

(2)

or $|f(x)| \in R(\alpha)$.

To P.T $\left|\int f\, d\alpha\right| \leq \int |f|\, d\alpha$.

choose $c = \pm 1$

So that

$$c\int_a^b f\, d\alpha \geq 0$$

consider, $\left|\int_a^b cf\, d\alpha\right| = \pm\int c\,f\,d\alpha$

$$\left|\int_a^b f\, d\alpha\right| = \pm c\int_a^b f\, d\alpha.$$

$$= \int_a^b cf\, d\alpha$$

$$\leq \int_a^b |cf|\, d\alpha$$ square inequality

$$\leq \int_a^b |c|\,|f|\, d\alpha$$

$$\leq c\int_a^b |f|\, d\alpha$$

$$\left|\int_a^b f\, d\alpha\right| \leq \int_a^b |f|\, d\alpha.$$

Definition:

The unit step function I defined by

$$f(x) = \begin{cases} 0 & ; x \leq 0 \\ 1 & x > 0 \end{cases}$$

Theorem : 6.15

If $a < s < b$ f is bounded on $[a,b]$ f is continuous at s and $\alpha(x) = I = (x-s)$ P.T

$\int_a^b f \, d\alpha = f(s)$.

Proof :

Consider the partition

$P = \{x_0 = a, x_1 = s, x_2 < x_3 = b\}$

$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$ $\quad \alpha x_i \quad \alpha x_{i-1}$

$= M_1 \Delta\alpha_1 + M_2 \Delta\alpha_2 + M_3 \Delta\alpha_3$

$= M_1 [\alpha(x_1) - \alpha(x_0)] + M_2 [\alpha(x_2) - \alpha(x_1)]$

$+ M_3 [\alpha(x_3) - \alpha(x_2)]$

$= M_1 (0-0) + M_2 (1-0) + M_3 (1-1)$.

$\Rightarrow U(P,f,\alpha) = M_2$.

M^{ly} $L(P,f,\alpha) = \sum_{i=1}^{3} m_i \Delta\alpha_i$.

$= m_2$.

$\alpha(x_1) = I(x_1 - s)$
$= 0$ for
$x_1 - s = 0$

Since f is continuous at s

$\therefore M_2$ & m_2 converges $f(s)$

as $x_0 \to x_1$

$f(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0. \end{cases}$

$\lim U(P,f,\alpha) = \int_a^b f \, d\alpha$

$= f(s)$

i.e: $\int_a^b f \, d\alpha = f(s)$.

Theorem 6.16 :

Let $c_n > 0$ for $1, 2 \ldots$ $\sum c_n$ converges $\{s_n\}$ is a sequense of distinct points in (a,b) and

$\alpha(x) = \sum_{n=1}^{\infty} c_n \, I(x - s_n)$. $I(x - s_n)$ is the unit step function p.T $\int_a^b f \, d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$.

Proof :

The series $\sum_{n=1}^{\infty} c_n I(x - s_n)$

cgs for $\sum c_n$ converges by data and $I(x - s_n) = 0$

or finite $\alpha(a) = \sum_{n=1}^{\infty} c_n I(a - s_n)$ for dat.

$$= \sum_{n=1} c_n \times 0 \quad \text{for } a < s_n \ \forall n,$$

and so $a - s_n < 0$

$\alpha(a) = 0$

$\alpha(b) = \sum_{n=1}^{\infty} c_n I(b - s_n)$ for data.

$$= \sum_{n=1}^{\infty} c_n \times 1 \quad \text{for } b > s_n \ \forall n \text{ and so}$$

$b - s_n > 0$ $\forall n$ and $I(b - s_n) = 1$

$\alpha(b) = \sum_{n=1}^{\infty} b \, c_n$

Let $\epsilon > 0$ be given.

choose $\epsilon > 0$ be N so that.

$\sum_{n+1}^{\infty} c_n < \epsilon \rightarrow (1)$ for $\sum c_n$ cgs by data

Let $\alpha_1(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$

$\alpha_2(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$

consider

$$\int_a^b f\,d\alpha_1 = \sum_{i=1}^{N} c_n I(x-s_n)$$

$$= \sum_{i=1}^{n} c_n f(s_n) \quad \text{by the } 6.15 \rightarrow ②$$

consider,

$$\alpha_2(b) - \alpha_2(a) = \sum_{n+1}^{\infty} c_n$$

$$< \epsilon \quad \text{by } ① \rightarrow ③$$

consider

$$\left| \int_a^b f\,d\alpha_2 \right| \leq M\left[\alpha_2(b) - \alpha_2(a)\right]$$

where $M = \sup M\, |f(x)|$ in $[a,b]$

$$\leq M\epsilon \text{ by } ③$$

consider $\int_a^b f\,d\alpha = \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2$

$$\leq \sum_{i=1}^{n} c_n f(s_n) + M\epsilon$$

or $\left| \int_a^b f\,d\alpha - \sum_{i=1}^{n} c_n f(s_n) \right| \leq M\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$

as $n \rightarrow \infty$

$$\int_a^b f\,d\alpha = \sum_{i=1}^{\infty} c_n f(s_n).$$

Theorem: 6.17.

$$\int_a^b f\,d\alpha = \int_a^b f(x)\alpha'(x)\,dx.$$

Proof :

Let $\epsilon > 0$ given by data $\alpha' \in R[a,b] \Rightarrow \exists$ a partition P. so that.

$$U(P,\alpha') - L(P,\alpha') < \epsilon \longrightarrow ①$$

consider.

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

$$= (x_i - x_{i-1})\, \alpha'(t_i) \text{ by [L.M.V Th of diff calculus]}$$

where $t_i, t \in \Delta x_i$

$$\Delta\alpha_i = \Delta x_i \cdot \alpha'(t_i) \longrightarrow ②$$

Again since $\alpha' \in R[a,b]$ we have.

$$\sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)|\, \Delta x_i < \epsilon \longrightarrow ③$$

consider

(or) $\sum_{i=1}^{n} |f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta x_i|$

$$= \left|\sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i - \sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta x_i\right|$$

by ② where $s_i \in \Delta x_i$.

Let $\sup|f(x)| \leq \min [a,b]$

$$\leq M \sum_{i=1}^{n} [\alpha'(s_i) - \alpha'(t_i)]\, \Delta x_i$$

$$\leq M\epsilon \text{ by } ③$$

or $|U(P,f,\alpha) - U(P,f\alpha')| \leq M\epsilon$

Taking infimum with in modulus we get,

$$\left|\int_a^b f\,d\alpha - \int_a^b (f\alpha')\,dx\right| \leq M\epsilon \to 0 \text{ as } \epsilon \to 0$$

④

$f \in R(\alpha)$ by data

$f\alpha' \in R[a,b]$

(A) becomes, $\int_a^b f\,d\alpha = \int f(\alpha')\,d\alpha$;

Theorem: 6.19 [Change of variables] (or) [Integrability by substitution]

Let ϕ be an increasing continuous function that maps $[A,B]$. Let α be monotonic increasing by $[a,b]$ Let $f \in R(\alpha)$ on $[a,b]$ Let B and g be defined on $[A,B]$ by $B(y) = \alpha[\phi(y)]$ & $g(y) = f[\phi(y)]$

P.T $\int_A^B g\,dB = \int_a^b f\,d\alpha$.

Proof:

Let $x = \phi(y)$

Corresponding to each partition P of $[a,b]$ $\exists$ a partition Q of $[A,B]$ since the values of f on $[x_{i-1}, x_i]$ are the same as the value of g on $[y_{i-1}, y_i]$

$\therefore U(Q,g,B) = U(P,f,\alpha)$

and $L(Q,g,B) = L(P,f,\alpha)$

We have,

$U(P,\alpha,\alpha) - L(P,\alpha,f) < \epsilon$

$\therefore U(P,f,\alpha) - L(P,f,\alpha) = U(Q,g,B) - L(Q,g,B) < \epsilon$

$\Rightarrow g \in R(B)$

$$\int_A^B g\,dB = \int_a^b f \cdot dx.$$

Theorem: Integration & Differentiation.

Integral function $F(x)$.

Let $f \in R[a,b]$ put $f(x) = \int_a^b f(t)dt$ P.T F is continuous on $[a,b]$. Also if F is continuous at $x_0 \in [a,b]$ P.T F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Proof:

Since $f \in R[a,b]$, f is bounded on $[a,b]$,

$\Rightarrow |f(t)| \leq m$ on $[a,b]$

$\Rightarrow |f(t)| \leq M$ on $[a,b]$

If $a \leq x \leq y \leq b$ then

consider.

$$|F(y) - F(x)| = \left| \int_a^y f\,dt - \int_a^x f\,dt \right| = \left| \int_x^y f\,dt \right|$$

$$\leq \int_a^y |f|dt$$

$$\leq M \int_a^y dt$$

$$\leq M (t)_x^y$$

$$\leq M (y-x)$$

$$\leq M(b-a)$$

$$< \epsilon \text{ if } |b-a| < \epsilon/M$$

i.e. if $|y-x| < \epsilon/M$

F(x) is continuous on [a, b]

Let $x_0 \in [a,b]$

we have proved that F(x) is continuous on [a, b] and so continuous at x_0. Let s, t $\in (x_0 - \delta, x_0 + \delta)$

consider $\left| \frac{f(t) - f(s)}{t-s} - f(x_0) \right|$

$$= \left| \frac{1}{t-s} \left[\int_s^t f(u)du - f(x_0) \right] \right|$$

$$= \left| \frac{1}{t-s} \int_s^t [f(u) - f(x_0)] du \right| \text{ for } \int_s^t f(x_0) du$$

$$< \left| \frac{1}{t-s} [f(u) - f(x_0)] \right|$$

$<$ for $|F(u) - F(x_0)| < \epsilon$ $(t-s)$ for

F is continuous at x_0

F is derivable at x_0 and $F'(x_0) = f(x_0)$.

Theorem: 6.21

State and proof the fundamental theorem of Integral calculus.

Imp

If $f \in R[a,b]$ and F is primitive of P.T

$$\int_a^b f(x) dx = F(b) - F(a).$$

Proof:

Let $\epsilon > 0$ be given.

Choose a partition $P = \{x_0 = a, x_1, \dots x_n = b\}$ of $[a,b]$ so that $U(P,f) - L(P,f) < \epsilon$ consider.

$F(x_i) - F(x_{i-1}) = f(t_i)(x_i - x_{i-1})$

$F(x_i) - F(x_{i-1}) = f(x_i)\Delta x_i$ [by M.V.T. of diff calculus where $t_i \in \Delta x_i$ for $F' = f$]

or $\sum_{i=1}^{n}[F(x_i) - F(x_{i-1})] = \sum_{i=1}^{n} f(t_i)\Delta x_i$

$[F(x_1) - F(x_0 = a) + F(x_2) - F(x_1) + F(x_3) - F(x_2) + \dots + F(x_n = b) - F(x_{n-1})] = \int_a^b f(x)dx$ as $n \to \infty$

$F(b) - F(a) = \sum_{i=1}^{n} f(x_i)\Delta x_i$

(or) $F(b) - F(a) = \int_a^b f(x)\,dx$.

Theorem: 6.22. [Integration by parts]

Let F and G be two differentiable function on $[a,b]$ Let $F' = f \in R$ and $G' = g \in R$ P.T

$$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx$$

Proof:

Let $H(x) = F(x)G(x)$

$H'(x) = F(x)g(x) + f(x)G(x) \to$ product rule of diff calculus

consider. $\int_a^b H'(x)dx = [H(x)]_a^b$

$= H(b) - H(a)$ (by fundamental theorem]

(or) $\int_a^b [F(x)g(x) + f(x)G(x)]\,dx$

$= H(b) - H(a)$.

(or) $\int_a^b F(x) g(x)\,dx = -\int_a^b f(x) G(x)\,dx + F(b) G(b) - F(a) G(a)$

$$\int_a^b F(x) g(x)\,dx = F(b) G(b) - F(a) G(a) - \int_a^b f(x) G(x)\,dx$$

Definition:

Integral of vector valued function.

Let $\vec{f}$ be a function on $[a,b] \subset R^1$ to R^k = [set of all k-tuples] $-(y_1, y_2 \dots y_k)$]

Eg:

$$\int_a^b \bar{f}\,d\alpha = [\int_a^b f_1\,d\alpha \int_a^b f_2\,d\alpha \int_a^b f_3\,d\alpha \dots \int_a^b f_k\,d\alpha] \in R^k -$$

$$\vec{f}(x) = [f_1(x), f_2(x) \dots f_k(x)] \in R^k.$$

Theorem: 6.24.

P.T fundamental theorem of integral calculus for $\vec{f}$ has also the same form as the fundamental theorem for f.

proof.

Let $\bar{f}$ and F maps $[a,b] \in R^1$ into R^k let $\vec{f} \in R(\alpha)$ and $\vec{F}' = \vec{f}$.

i.e, $\vec{F}$ is primitive of $\vec{f}$

the fundamental theorem of integral calculus is true for every component f_i of $\vec{f}$

$$\int_a^b \vec{f}\, dx = \left[\int_a^b f\, d\alpha \cdot \int_a^b f_2\, d\alpha - - - \int_a^b f_k\, d\alpha \right]$$

$$[F_1(b) - F_1(a), F_2(b) - F_2(a), F_i(b) - F_i(a),$$
$$F_k(b) - F_k(a)]$$

where F_i is primitive of f_i Δi

$$\int_a^b \vec{f}\, d\alpha = \vec{F}(b) - \vec{F}(a)$$

Highest coefficient of x is 1 is called primitive Ex: x^n

The fundamental theorem has same form for $\vec{f}$ has for f.

Theorem: 6.25

If f maps $[a,b]$ of R' into R^k and $\vec{f} \in R(\alpha)$ for some function α. P.T $\left| \int_a^b \vec{f}\, d\alpha \right| \le \int_a^b |\vec{f}|\, d\alpha$.

Proof:

Let $f_1, f_2 - - - f_k$ be the component of $\vec{f}$

$\Rightarrow |\vec{f}| = \sqrt{f_1^2 + f_2^2 + - - - + f_k^2}$

If $\vec{f} \in R(\alpha)$ then $f_1, f_2 - - - f_k$ all $\in R(\alpha)$

$\Rightarrow f_1^2, f_2^2 - - - f_k^2$ all $\in R(\alpha)$

$\Rightarrow f_1^2 + f_2^2 + - - - + f_k^2$ all $\in R(\alpha) \Rightarrow |\vec{f}| \in R(\alpha)$.

To prove that $\left| \int_a^b \vec{f}\, d\alpha \right| \le \int_a^b |\vec{f}|\, d\alpha$.

put $\vec{y} = \int_a^b \vec{f}\, d\alpha = (y_1, - - - y_n)$ where.

$$y_i = \int_a^b f_i\, d\alpha.$$

$$|\vec{y}| = \sum_{j=1}^{n} y_j^2 = \sum_{j=1}^{n} y_j y_j$$

$$|\vec{y}|^2 = \sum_{j=1}^{n} y_j^2 = \sum_{j=1}^{n} y_j y_j$$

(or)

$$|\vec{y}|^2 = \sum_{i=1}^{n} y_i \int_a^b f_i \, d\alpha.$$

$$\int_a^b \left(\sum_{j=1}^{m} y_j \right) f_j \, d\alpha$$

$$\leq \int_a^b |\vec{y}| \, |\vec{f}| \, d\alpha$$

$$|\vec{y}|^2 \leq |\vec{y}| \int_a^b |\vec{f}| \, d\alpha$$

$$|\vec{y}| \leq \int_a^b |f| \, d\alpha$$

(or) $$\left| \int_a^b f \, d\alpha \right| \leq \int_a^b |\vec{f}| \, d\alpha$$

i.e., modulus of the integral $\leq$ integral of the modulus.

Rectifiable curves:

Definition:

A Rectifiable curve is one whose length can be measured.

Definition of curve in R^k.

A continuous map γ of our interval $[a,b]$ of R^1 into R^k is called curve in R^k.

Let $f \in [a,b]$

$$\gamma(t) = (\gamma_1(t), \gamma_2(t) \ldots \gamma_k(t)) \in R^k.$$

Thus $\gamma(t)$ is a point in R^k which has n co-ordinates i.e., $(\gamma_1(t) \ldots \gamma_n(t))$

As t varies over $[a,b]$ we get different continuous point in R^k.

If γ is 1-1 the curve is an arc.

If $\gamma(a) = \gamma(b)$ then γ is a closed curve. $\times$

Definition of the polynomial length of γ

i.e. $\lambda(P,\gamma)$

Let P be a partition of $[a,b]$

i.e., $P = [x_0 = a, x_1, x_2 \ldots x_{i-1}, x_i \ldots x_n = b]$

Now $\gamma(x_0), \gamma(x_1) \ldots (\gamma(x_{i-1}), \gamma(x_i))$ are all points in R^k

Join the pts by segments of straight lines

The ith interval Δx_i maps into the line segment.

$$|\lambda(x_{i-1}), \lambda(x_i)|$$

Now,

$$\sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|$$

= Sum of all line segment of γ

= polynomial length of γ

$= \lambda(P,\gamma)$

If the partition P becomes finer and finer the polygon becomes a curve and $\lambda(P,\gamma)$ becomes the length of the curve $\gamma = \lambda(\gamma)$

i.e: $\lambda(P,\gamma) \to \lambda(\gamma)$. If $\lambda(\gamma)$ is the fine finite. i.e: $< +\infty$ then the curve r is said to be rectifiable.

Theorem: 6.27.

If γ' is continuous on $[a,b]$. P.T γ is rectifiable and $\Lambda(\gamma) = \int_a^b |\gamma'(t)|\, dt$.

V.I.

Imp.

LOH

proof:

If $a \le x_{i-1} \le x_i \le b$

then $|\gamma(x_i) - \gamma(x_{i-1})| = \left| \int_{x_{i-1}}^{x_i} \gamma'(t)\, dt \right|$

By fundamental theorem

$$\le \int_{x_{i-1}}^{x_i} |\gamma'(t)|\, dt$$

or $\sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})| \le \sum_{i=1}^{n} |\gamma'(t)|\, dt$

(length of the polygon)

(or) $\Lambda(P,\gamma) \le \int_a^b |\gamma'(t)|\, dt$

In the limit as $n \to \infty$ we get,

$$\Lambda(\gamma) \le \int_a^b |\gamma'(t)|\, dt \to (1)$$

To prove the opposite inequalities. Let $\epsilon > 0$ be given,

Since γ' is uniformly continuous on $[a,b]$

$\therefore \delta > 0$ so that.

$$|\gamma'(s) - \gamma'(t)| < \epsilon \text{ if } |s-t| < \delta \to (2)$$

If $P = \{x_0, \ldots x_n\}$ be the partition of $[a,b]$

with $\Delta x_i < \delta, \forall i$

If $t \in \Delta x_i$, we get

$|\gamma'(t)| \leq |\gamma'(x_i)| + \epsilon$ from ②

(or) $$\int_{x_{i-1}}^{x_i} |\gamma'(t)| dt \leq \int_{x_{i-1}}^{x_i} |\gamma'(x_i)| dt + \epsilon \int_{x_{i-1}}^{x_i} dt$$

$$\leq |\gamma'(x_i)| \Delta x_i + \epsilon \Delta x_i$$

$$\leq \int_{x_{i-1}}^{x_i} |\gamma'(x_i) + \gamma'(t) - \gamma'(t)| dt + \epsilon \Delta x_i$$

$$\leq \int_{x_{i-1}}^{x_i} |\gamma'(t)| dt + \int_{x_{i-1}}^{x_i} |\gamma'(x_i) - \gamma'(t)| dt + \epsilon \Delta x_i$$

$$\int_{x_{i-1}}^{x_i} |\gamma'(t)| dt \leq [\gamma(x_i) - \gamma(x_{i-1})] + \epsilon \Delta x_i + \Delta x_i$$

Adding we get,

$$\int_a^b |\gamma(t)| dt \leq \Lambda(P, \gamma) + 2\epsilon (b-a)$$

$$\leq \Lambda(\gamma) + 2\epsilon (b-a)$$

$$\leq \Lambda(\gamma) \rightarrow ③.$$

From ② & ③ we get,

$$\int_a^b |\gamma'(t)| dt = \Lambda(\gamma)$$

Hence the proof.