$p < \frac{f(x)}{g(x)} < q$

$\frac{f(x)}{g(x)}$ lies between p & q already we know that

A lies between p & q ~~also~~

$\frac{f(x)}{g(x)} \to A$ as $x \to a$.

UNIT – III

Recall Rieman of integral

Definition :

Let $[a,b]$ be the given integral by a partition p of $[a,b]$ we mean a finite set of points.

$(x_0, x_1, \ldots x_n)$

where $a = x_0 \le x_1 \le \ldots \le x_n = b$

Let $[x_{r-1}, x_r]$ be the r^{th} sub integral where

$\Delta x_r = x_r - x_{r-1}$

$= \text{length of } \Delta x_r$

Let $M_r = \sup m\ f(x)$ in Δx_r

$m_r = \inf m\ f(x)$ in Δx_r

consider the sum $= \sum_{r=1}^{n} M_r \Delta x_r$

= upper sum of corresponding to the partition n

$U(P, f, \alpha)$

consider the sum $= \sum_{r=1}^{m} m_r \Delta x_r$

= lower sum of the correspond to the partition.

$= L(P,f)$

Inf $(U(P,f))$ is called $\overline{\int_a^b} f\,dx$ upper R.I

Sup $L(P,f)$ is called $\underline{\int_a^b} f\,dx$ lower R.I

Over all partition.

If $\overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx$ then f is said to be Reimann integral

i.e., $f \in R \Rightarrow$ The set of all Reiman integrable function and their common value is put as $\int_a^b f\,dx$.

Since f is bounded on $[a,b]$ therefore lower and upper bounds m & M for f on $[a,b]$.

So that $m \leq f(x) \leq M$

(or) $\sum_{i=1}^{n} m\,\Delta x_r \leq \sum_{i=1}^{n} f(x)\,\Delta x_r \leq \sum_{i=1}^{n} M\,\Delta x_r$

(or)

$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$

$L(P,f)$ & $U(P,f)$ are bounded.

Riemann Stieltjes integral:

Let $\alpha(x)$ be a monotonically increasing function.

i.e: increasing on $[a,b]$

Since $\alpha(a)$ & $\alpha(b)$ are finite

$\alpha(x)$ is bounded on $[a,b]$

Let P be a partition of $[a,b]$

Let $[x_{i-1}, x_i]$ be the ith interval of P.

Let $[\alpha(x_{i-1}), \alpha(x_i)]$ be called $\Delta\alpha_i$ length of

$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$

$\Delta\alpha_i \geq 0$ for α.

Now the upper sum.

$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$ is called the upper sum of f corresponding to P.

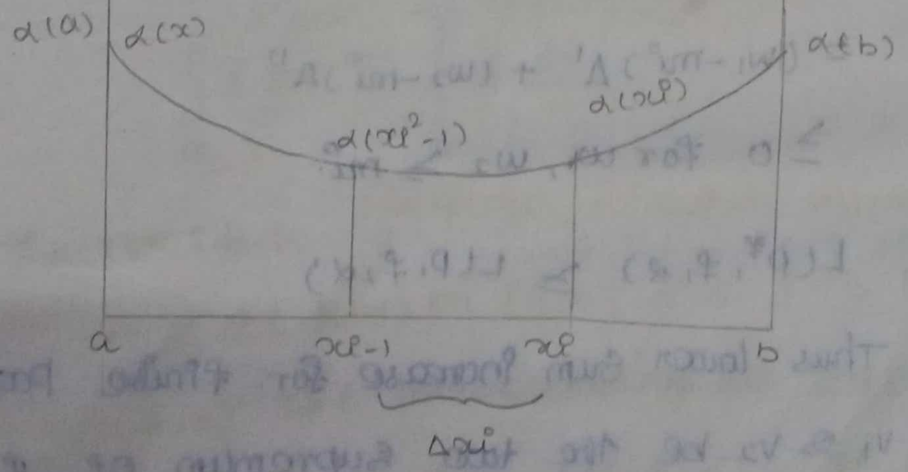
$L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$ is called the lower sum of f corresponding to P.

where M_i & m_i are supremum & infimum of $f(x)$ in $\Delta\alpha_i$

Inf M $U(P,f,\alpha)$ is called as $\overline{\int_a^b} f\, d\alpha$.

Sup m $L(P,f,\alpha)$ is called as $\underline{\int_a^b} f\, d\alpha$

(over all partition) the lower integral of f.



Theorem: 6.4

If P^* is a refinement of P. P.T

$L(P^*, f, \alpha) \geq L(P, f, \alpha)$

(or)

$L(P, f, \alpha) \leq L(P^*, f, \alpha)$

and

$U(P^*, f, \alpha) \leq U(P, f, \alpha)$

Proof:

Let P^* be the refinement of P by the addition of single point x^* in P. Let w_1 & w_2 be the two infimums of $f(x)$ in $(x_{i-1}, x^*) = \Delta'$ & $(x^*, x_i) = \Delta''$

Let m_i be the infimum of f in Δx_i clearly

$w_1 \geq m_i$ & $w_2 \geq m_i$.

Consider,

$L(P^*, f, \alpha) - L(P, f, \alpha) = (w_1\Delta' + w_2\Delta'') - m_i \Delta x_i$

$= w_1(\Delta') + w_2(\Delta'') - m_i(\Delta' + \Delta'')$

$= (w_1 - m_i)\Delta' + (w_2 - m_i)\Delta''$

≥ 0 for $w_1, w_2 \geq m_i$

$L(P^*, f, \alpha) \geq L(P, f, \alpha)$

Thus lower sum increase for finite partitions

Let v_1 & v_2 be the two supremum of $f(x)$ in Δ' & Δ'' $U(P, f, \alpha) - U(P^*, f, \alpha)$

$= (M_i\Delta' + M_i\Delta'') - (v_1\Delta' + v_2\Delta'')$

$= (M_i - v_1)\Delta' + (M_i - v_2)\Delta''$

≥ 0 for $M_i \geq U_1, U_2$

$U(P, d, \alpha) \geq U(P^*, d, \alpha)$

upper sum diminishes for finite partitions.

Theorem 6.5

P.T $\underline{\int_a^b} f\, d\alpha \leq \overline{\int_a^b} f\, d\alpha$.

Proof:

Let P_1 and P_2 be two partitions of $[a,b]$ let $P^* = P_1 \cup P_2$ = a refinement of both P_1 & P_2 consider

$L(P_1, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P_2, f, \alpha)$

$L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$

Taking supremum of L.H.S over all partitions keeping R.H.S. fixed. In the limit we get.

$\underline{\int_a^b} f\, d\alpha \leq U(P_2, f, \alpha)$

keeping L.H.S as fixed and take infimum of R.H.S over all partitions. In the limit t, we get,

$\underline{\int_a^b} f\, d\alpha \leq \overline{\int_a^b} f\, d\alpha$.

Theorem: 6.6.

P.T $f \in R(\alpha)$ on $[a,b]$ i.e. f is R.S integrable on $[a,b]$ iff corresponding to $\epsilon > 0$ $\exists$ a partition P. so that $U(P, f, \alpha) - L(P, f, \alpha) \rightarrow$ (1).

Proof:

Case (i):

Let (1) be true

For every P we have

$$L(P,f,\alpha) \le \underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha \le U(P,f,\alpha) \rightarrow ②$$

By ① $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$

Putting this in ② we get

$$\overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha < \epsilon \longrightarrow ③$$

Let $\epsilon \to 0$ as ϵ is arbitrary from ③ we get

$$\overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha = 0$$

or $\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$

$\Rightarrow f \in R(\alpha)$

① is necessary condition for R.S. integrability

Case (ii):

Let $f \in R(\alpha) \Rightarrow \int_a^b f\,d\alpha$ exists corresponding to $\epsilon > 0$ $\exists$ a partition P_1 & P_2 so that,

$$U(P_2,f,\alpha) - \int_a^b f\,d\alpha < \epsilon/2 \longrightarrow ④$$

$$\int_a^b f\,d\alpha - L(P_1,f,\alpha) < \epsilon/2 \longrightarrow ⑤$$

④+⑤.

$$U(P_2,f,\alpha) - L(P_1,f,\alpha) < \epsilon \longrightarrow ⑥$$

Let $P^* = P_1 \cup P_2$ = refinement of P_1 & P_2

From (6), (4) & (5) we get,

$$U(P^*, f, \alpha) \le U(P_3, f, \alpha) < \int_a^b f \, d\alpha + \epsilon/2$$

$$< L(P, f, \alpha) + \epsilon/2 + \epsilon/2$$

$$< L(P, f, \alpha) + \epsilon$$

$$< L(P^*, f, \alpha) + \epsilon$$

$$U(P^*, f, \alpha) < L(P^*, f, \alpha) + \epsilon$$

(or) $U(P^*, f, \alpha) - L(P^*, f, \alpha) < \epsilon$

Theorem: 6.7.

Let $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon \rightarrow (1)$ for some ϵ and some corresponding P. Let $P = \{x_0, \ldots, x_n\}$ Let s_i and $t_i \in \Delta x_i$ of P. P.T. b) $\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta \alpha_i < \epsilon$

a) (1) is for any refinement of P.

c) If $f \in R(\alpha)$. P.T $\left| \sum_{i=1}^{n} f(t_i) \Delta \alpha_i - \int_a^b f \, d\alpha \right| < \epsilon$

Proof:

a) Let $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon \longrightarrow (1)$

Let P^* be a refinement of P we have,

$$U(P^*, f, \alpha) \le U(P, f, \alpha)$$

$$L(P^*, f, \alpha) \ge L(P, f, \alpha)$$

$$-L(P^*, f, \alpha) \le -L(P, f, \alpha) \longrightarrow (2)$$

(1) + (2) $U(P^*, f, \alpha) - L(P^*, f, \alpha) \le U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$

by (1).

b) we have,

$|f(s_i) - f(t_i)| \le M_i - m_i$ where M_i and m_i are sup M and inf M of $f(x)$ in Δx_i

(s_i & t_i $\in \Delta x_i$)

Both side take

(or) $\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i \leq \sum_{i=1}^{n} (M_i - m_i) \Delta\alpha_i$

$\leq \sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i$

$\leq U(P, f, \alpha) - L(P, f, \alpha)$

$< \epsilon$ by (1),

c) we have,

$L(P, f, \alpha) \leq \sum_{i=1}^{n} f(t_i) \Delta\alpha_i \leq U(P, f, \alpha)$

(or) in the limit we get,

$L(P, f, \alpha) \leq \int_a^b f . d\alpha \leq U(P, f, \alpha)$

$f \in R(\alpha) \Rightarrow U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$

Let $t_i \in \Delta x_i$

To P.T $\left| \sum_{i=1}^{n} f(t_i) \Delta x_i - \int_a^b f \, d\alpha \right| \leq \epsilon$

Let M_i and m_i be the bounds of $f(x)$ is $\Delta\alpha_i$

$\Rightarrow m_i \leq f(t_i) \leq M_i$

$\Rightarrow \sum_{i=1}^{m} m_i \Delta\alpha_i \leq \sum_{i=1}^{n} f(t_i) \Delta\alpha_i \leq \sum_{i=1}^{n} M_i \Delta\alpha_i$

$\Rightarrow L(P, f, \alpha) \leq \sum_{i=1}^{n} f(t_i) \Delta\alpha_i \leq U(P, f, \alpha) \rightarrow (1)$

W.K.T

$L(P, f, \alpha) \leq \int_a^b f \, d\alpha \leq U(f, P, \alpha) \rightarrow (2)$

From ① we get,

$\sum_{i=1}^{n} f(t_i)\Delta\alpha_i$ lies between $L(P,f,\alpha)$ & $U(P,f,\alpha)$

From ② we get.

$\int_a^b f\,d\alpha$ lies between $L(P,f,\alpha)$ and $U(P,f,\alpha)$ By data

$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$

we get,

$$\left| \sum f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha \right| < \epsilon$$

Theorem: 6.8

SM 2011

If f is continuous on $[a,b]$ P.T $f \in \mathcal{R}(\alpha)$ in $[a,b]$

Proof:

Let $\epsilon > 0$ be given choosen $\eta > 0$ so that.

$$\eta < \frac{\epsilon}{\alpha(b)-\alpha(a)} \longrightarrow \text{I}$$

since f is uniformly continuous on $[a,b]$ $\exists$ a $\delta > 0$

so that $|f(x) - f(t)| < \eta$

if $x \in [a,b]$ & $|x-t| < \delta$, $t \in [a,b]$

Let P be a partition. so that $\Delta x_i < \delta$

$\Rightarrow M_i - m_i \leq \eta$

$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha)$

$= \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i$

$\leq \sum_{i=1}^{n} \eta\,\Delta\alpha_i$

$< \eta \sum_{i=1}^{n} \Delta\alpha_i$

$< \eta \sum_{i=1}^{n} \Delta \alpha_i$

$< \eta [\alpha(b) - \alpha(a)] \cdot [\cancel{\alpha(b) - \alpha(a)}]$

$< \frac{\epsilon}{\cancel{\alpha(b) - \alpha(a)}} [\cancel{\alpha(b) - \alpha(a)}]$

$< \epsilon$. $f \in R(\alpha)$ in $[a,b]$

Theorem: 6.9

2/2 If f is monotonic in $[a,b]$ p.T $f \in R(\alpha)$ on $[a,b]$ α is continuous on $[a,b]$

Proof:

Let $\epsilon > 0$ be given choose a partition P. so that, $\Delta \alpha_i = \frac{\alpha(b) - \alpha(a)}{n} \rightarrow (1)$.

This is possible for α is continuous on $[a,b]$

Let $f(x)$ be.

$\Rightarrow M_i = f(x_i)$ & $m_i = f(x_{i-1})$.

consider,

$U(f,P,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta \alpha_i - \sum_{i=1}^{n} m_i \Delta \alpha_i$

$= \sum_{i=1}^{n} f(x_i) \Delta \alpha_i - \sum_{i=1}^{n} f(x_{i-1}) \Delta \alpha_i$

$= \Delta \alpha_i \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]$

$= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]$

$= \frac{\alpha(b) - \alpha(a)}{n}$

$$= \frac{\alpha(b)-\alpha(a)}{n} [f(x_1) - f(x_0) + (f(x_2) - f(x_1)) + \cdots + f(x_n) - f(x_{n-1})]$$

$$= \frac{\alpha(b)-\alpha(a)}{n} [f(x_n) - f(x_0)]$$

$$= \frac{\alpha(b)-\alpha(a)}{n} \times f(b) - f(a).$$

$$= \frac{\text{a finite value}}{n} \Rightarrow 0 \text{ as } n \to \infty$$

$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$ where $\epsilon \to 0$

$f \in R(\alpha)$ in $[a, b]$

Similarly we can p.t $f \in R(\alpha)$ in $[a, b]$ if f is in $[a, b]$

Theorem: 6.10

Let f be bounded on $[a, b]$. Let f be any finite many points of discontinuity on $[a, b]$. Let α be continuous at every points at which f is discontinuous P.T $f \in R(\alpha)$.

Proof:

Let $\epsilon > 0$ be given. Let M = supremum of $f(x)$ in $[a, b]$. Let E be the set of points $(P_1, \cdots P_n)$ where f is discontinuous at all of E, we can cover E by finitely many disjoint intervals $[u_i, v_i]$ so that the sum of the corresponding differences

$\alpha(v_j) - \alpha(u_j)$ is $<$ than ϵ. The sum of the length of those intervals $< \epsilon \to (1)$

we can place the intervals. so that every point $[a, b]$ of descontinity of f in one of $[u_i, v_i]$'s Remove all the segments $[u_j, v_j]$ from $[a, b]$ the remaining set K is compact.

i.e. bounded and closed. Now f is uniformly continuous on k. $\exists\ \delta > 0$ so that.

$|f(s) - f(t)| < \epsilon$ if $s, t \in k$ & $|s-t| < \delta$

From a partition $P = [x_0, x_1, \dots x_n]$ of $[a,b]$.

as follows each $u_j, v_j \in P$ and no point of $(u_j, v_j) \in P$

$M_i - m_i \leq \epsilon$ is every interval $\in k$ where f is continuous and $M_i - m_i \leq 2M$ in every interval where f has a discontinuous point,

consider,

$$U(P, f, \alpha) - L(P, f, \alpha)$$

$$= \sum_{i=1}^{n_1} (M_i - m_i)\Delta\alpha_i + \sum_{j=1}^{n_2} (M_i - m_i)\Delta\alpha_i$$

where the first summation is over the intervals where f is continuous and the 2nd summation is over all intervals containing discontinuous points of f.

Now $U(P, f, \alpha) - L(f, P, \alpha)$

$$\leq \sum_{i=1}^{n_1} \Delta\alpha_i + 2M \sum_{i=1}^{n_2} \Delta\alpha_i$$

$$\leq \epsilon [\alpha(b) - \alpha(a)] + 2M \sum_{i=1}^{n_2} \Delta\alpha_i$$

$$\leq \epsilon [\alpha(b) - \alpha(a)] + 2M\epsilon$$

$$\leq \epsilon [\alpha(b) - \alpha(a)] + 2M] \text{ finite}$$

$< \epsilon$

$\epsilon \to 0$

$f \in R(\alpha)$.

Theorem: 6.11

Let $f \in R(\alpha)$ on $[a,b]$. Let ϕ be continuous on $[M,m]$. Let $h(x) = \phi f(x)$ on $[a,b]$ P.T $h \in R(\alpha)$ m, M be the bounds of $f(x)$ on $[a,b]$; Domain of f is $[\alpha(a), \alpha(b)]$ Domain of ϕ is $[M,m]$

Proof:

choose $\epsilon > 0$, since ϕ is uniformly continuous on $[m,M]$ $\exists$ $\delta > 0$ so that

$|\phi(s) - \phi(t)| < \epsilon$ if $|s-t| < \delta$

Since $f \in R(\alpha)$ $\exists$ a partition $P = \{x_0, x_1, \dots x_n\}$ of $[a,b]$ so that $U(f,P,\alpha) - L(f,P,\alpha) < \delta^2 \rightarrow (1)$

Let m_i & M_i be the bounds of $f(x)$ in Δx_i. Let m_i^* & M_i^* be the bounds of $h(x)$ in Δx_i

Divide the number of $1, 2 \dots n$ into 2 classes A & B.

If $i \in A$, Let $M_i - m_i < \delta < \epsilon \rightarrow (2)$

if $i \in B$, Let $M_i - m_i \geq \delta \rightarrow (3)$

For $i \in A$ we have $M_i - m_i < \delta < \epsilon$

$\Rightarrow M_i^* - m_i^* \leq \epsilon$

where $M = \sup$ of $|\phi(x)| \rightarrow (4)$

From (3) we have $\delta \geq M_i - m_i$

$$\delta \sum_{i=B} \Delta \alpha_i \leq \sum_{i \in B} (M_i - m_i) \Delta \alpha_i$$

(or) $\delta \sum \Delta \alpha_i < U(P,f,\alpha) - L(P,f,\alpha) < \delta^2$ by (1)

(or) $\sum_{i \in B} \Delta \alpha_i < \delta < \epsilon \rightarrow (5)$.

consider $U(P, f, \alpha) - L(P, f, \alpha)$

$$= \sum_{i \in A} (M_i^* - m_i^*) \Delta \alpha_i + \sum_{i \in B} (M_i^* - m_i^*) \Delta \alpha_i$$

$$\leq \epsilon \, (\alpha(b) - \alpha(a)) + 2M \sum_{i \in B} \Delta \alpha_i$$

by ②.

$$\leq \epsilon [\alpha(b) - \alpha(a)] + 2M\epsilon \rightarrow ⑤$$

$$< \epsilon [\alpha(b) - \alpha(a) + 2M]$$

$\Rightarrow 0$ as $\epsilon \rightarrow 0$

$\Rightarrow h(x) \in R(\alpha)$

$h \in R(\alpha)$.

Properties of the R.S Integral.

Theorem : 6.12.

i) If $f_1 \in R(\alpha)$ and $f_2 \in R(\alpha)$ P.T $f_1 + f_2 \in R(\alpha)$.

Proof:

we have $\sup(f_1 + f_2) \leq \sup f_1 + \sup f_2$

and $\inf(f_1 + f_2) \geq \inf f_1 + \inf f_2$

since $f_2 \in R(\alpha)$

$\therefore U(P_2, f_2, \alpha) - L(P_2, f_2, \alpha) \leq \epsilon$

for a given ϵ and corresponding to partition

$P_2 \rightarrow$ ①.

Let $P = P_1 \cup P_2$. a refinement of both P_1 & P_2

we know that the inequality (I) are true for the finite partition P.

we get $U(P, f_1, \alpha) - L(P, f_1, \alpha) < \epsilon$ &

$U(P, f_2, \alpha) - L(P, f_2, \alpha) < \epsilon$

Adding we get,

$U(P, f_1+f_2, \alpha) - L(P, f_1+f_2, \alpha) < 2\epsilon \to 0$ as $\epsilon \to 0$

$\Rightarrow f_1+f_2 = f \in R(\alpha)$.

ii) To show that $\int_a^b (f_1+f_2)\,d\alpha = \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha$.

Proof:

we have

$U(P, f, \alpha) < \int_a^b f_1\,d\alpha + \epsilon \to$ a result.

& $U(P, f_2, \alpha) < \int_a^b f_2\,d\alpha + \epsilon \to$ a result

Adding we get,

$U(P, f_1+f_2, \alpha) < \int_a^b f\,d\alpha$ for f_1+f_2.

$\leq U(P, f_1, \alpha) + U(P, f_2, \alpha)$ for

$\sup f_1+f_2 \leq \sup f_1 + \sup f_2$

$\leq \int_a^b f_1\,d\alpha + \epsilon + \int_a^b f_2\,d\alpha + \epsilon$

$\int_a^b (f_1+f_2)\,d\alpha \leq \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha + 2\epsilon \to (2)$

write $-f_1$ for f_1 and $-f_2$ for f_2 in (2) we get,

$\int_a^b (-f_1-f_2)\,d\alpha \leq -\int_a^b f_1\,d\alpha - \int_a^b f\,d\alpha$.

or) $-\int_a^b (f_1+f_2)\,d\alpha \leq -\int_a^b f_1\,d\alpha - \int_a^b f_2\,d\alpha$.

or) $\int_a^b (f_1+f_2)\,d\alpha \geq \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha \to (3)$

from ② & ③ we get.

$$\int_a^b (f_1+f_2)\,d\alpha = \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha.$$

iii) If $f \in R(\alpha)$ P.T $(cf) \in R(\alpha)$ where c is a constant and S.T $\int_a^b cf\,dx = c\int_a^b f\,dx$.

proof:

since $f \in R(\alpha)$

$$\int_a^b f\,d\alpha = \lim_{n\to\infty} U(f, P_n, \alpha) \text{ where } P_n \text{ is a sequence}$$

of finer and finer partition consider.

$$\int_a^b cf\,d\alpha = \lim_{n\to\infty} U(cf, P_n, \alpha)$$

$$= c \lim_{n\to\infty} U(f, P_n, \alpha)$$

$$= c\int_a^b f\,d\alpha.$$

b) If $f_1(x) \le f_2(x)$ on $[a,b]$ P.T $\int_a^b f_1\,d\alpha \le \int_a^b f_2\,d\alpha$.

proof:

$$\int_a^b f_1\,d\alpha = \lim_{n\to\infty} U(P_n, f, \alpha)$$ [where P_n is a sequence of finer and finer partition]

$$= \lim_{n\to\infty} \sum_{i=1}^{n} m_i\,d\alpha \text{ where } M_i = \sup \text{ of } f_1 \text{ in } \Delta x_i$$

$$\le \lim_{n\to\infty} \sum_{i=1}^{n} M_i\,d\alpha \text{ where } M_i' = \sup \text{ of } f_2 \text{ in } \Delta x_i,$$

for $M' \le M$ by data.