11

Proof :

Let $\epsilon > 0$ be given,

Since $\{f_n\} \to f$ uniformly, it satisfies

Cauchy's criterion of convergence.

$\therefore$ There exists N such that $n \geq N$,

$m \geq N$ $t \in E$.

$\Rightarrow |f_n(t) - f_m(t)| \leq \epsilon \rightarrow (1)$

Letting $t \to x$ in (1), we get

$$\left| \lim_{t \to x} f_n(t) - \lim_{t \to x} f_m(t) \right| \leq \epsilon.$$

$\Rightarrow |A_n - A_m| \leq \epsilon$, for $n \geq N$, $m \geq N$.

$\therefore \{A_n\}$ is a cauchy sequence and

hence converges to say A.

Next, $|f(t) - A| \leq |f(t) - f_n(t) + f_n(t) - A_n + A_n - A|$

$\leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A| \rightarrow (2)$

we first choose n such that,

$|f(t) - f_n(t)| \leq \epsilon/3 \rightarrow (3)$

for all $t \in E$ (since $\{f_n\}$ is uniform convergence)

Since $\lim_{n \to \infty} A_n = A$

$\therefore |A_n - A| \leq \epsilon/3 \rightarrow (4)$

Then for this n, choose a nhd V of x such that $|f_n(t) - A_n| \leq \epsilon/3 \rightarrow (5)$

(if $t \in V \cap E$ and $t \neq x$).

using (3), (4) & (5) in (2) we get

$$|f(t) - A| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}.$$

$\leq \epsilon$ provided $t \in V \cap E$ and $t \neq x$

(i.e) $\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n$.

Theorem 5:

If $\{f_n\}$ is a sequence of continuous function on E and if $f_n \to f$ uniformly on E Then f is continuous on E.

Proof:

Since $\{f_n\}$ is a sequence of continuous functions,

$$\therefore \lim_{t \to x} f_n(t) = f_n(x) \rightarrow (1)$$

Suppose $f_n \to f$ uniformly on E, we have $\lim_{n \to \infty} f_n(t) = f(t) \rightarrow (2)$.

and by theorem (4) we have

$$\lim_{t \to \infty} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t)$$

$\Rightarrow \underset{t \to x}{Lt} f(t) = \underset{n \to \infty}{Lt} f_n(x)$ [by (5) & (1)]

$\Rightarrow \underset{t \to x}{Lt} f(t) = f(x)$ [by (2)].

Hence f is continuous on E.

Note:

The converse of the above theorem is not true.

(i.e) The sequence of continuous function may converge to continuous but the convergence need not be uniform.

[proof is Ex (5)] (Book Ex 7.6).

Theorem 6:

Suppose K is compact and

(a) $\{f_n\}$ is a sequence of continuous functions on K.

(b) $\{f_n\}$ converge pointwise to a continuous function f on K.

(c) $f_n(x) \geq f_{n+1}(x)$ $\forall x \in K$, $n = 1, 2 \ldots$

Then $f_n \to f$ uniformly on K.

Proof:

put $g_n = f_n - f$

since $\{f_n\}$ is continuous and converges to continuous function f,

$\therefore g_n$ is also continuous.

And $g_n \to 0$ pointwise and $g_n \geq g_{n+1}$, Now we have to prove that $g_n \to 0$ uniformly

Let $\epsilon > 0$,

Let $K_n = \{x / x \in K \text{ and } g_n(x) \geq \epsilon\}$

Since g_n is continuous.

By known theorem, [Book 4.8]

"A mapping f of a metric space X into a metric space Y is continuous on X iff $f^{-1}(V)$ is open in X for every open set V in Y".

K_n is closed.

Since $K_n \subseteq K$ $\quad\therefore K_n$ is also compact.

Since $g_n \geq g_{n+1}$ $\quad K_n \supset K_{n+1}$.

Fix $x \in K$,

Since $g_n(x) \to 0$ pointwise,

we see that $x \notin K_n$, if n is sufficiently large. Thus $x \notin \bigcap K_n$

$\Rightarrow \bigcap K_n = \phi$.

Hence K_N is empty for some N.

$\Rightarrow 0 \leq g_n(x) < \epsilon$ for all $x \in K$ and $\forall\, n \geq N$.

(i.e) $g_n(x)$ converges '0' uniformly.

$\Rightarrow g_n(x) \to 0$ uniformly.

$\Rightarrow f_n - f \to 0$ uniformly.

$\Rightarrow f_n \to f$ uniformly.

Defn:

If X is a metric space, $\mathcal{C}(X)$ will denote the set of all complex valued continuous bounded functions with domain X.

Note:

$\mathcal{C}(X)$ consists of all complex continuous functions on X if X is compact.

Each $f \in \mathcal{C}(X)$, its supremum norm

$$\|f\| = \sup_{x \in X} |f(x)|.$$

Since f is assumed to be bounded, $\|f\| < \infty$.

Clearly (i) $\|f\| = 0 \Leftrightarrow f(x) = 0 \ \forall x \in X$ i.e., if $f = 0$

(ii) If $h = f+g$ then

$$|h(x)| \leq |f(x)| + |g(x)| \leq \|f\| + \|g\|.$$

for all $x \in X$. Hence $\|f+g\| \leq \|f\| + \|g\|$.

If we define $d(f,g) = \|f-g\|$, $f,g \in \mathcal{C}(X)$

Then $(\mathcal{C}(X), d)$ satisfies Axioms of metric space.

Hence $\mathcal{C}(X)$ is a metric space.

Defn: A sequence $\{f_n\}$ converges to f with respect to the metric of $\mathscr{C}(X)$ iff $f_n \to f$ uniformly on X.

Defn: The closed subsets of $\mathscr{C}(X)$ are called uniformly closed and the closure of a set $A \subset \mathscr{C}(X)$ is called uniform closure.

Theorem 7:

If $\mathscr{C}(X)$ is a set of all complex valued continuous bounded functions with domain X and the metric in $\mathscr{C}(X)$ is defined as $d(f,g) = \|f-g\|$ Then P.T $(\mathscr{C}(X), d)$ is a complete metric space.

Proof:

To prove $(\mathscr{C}(X), d)$ is complete, we have to prove that every Cauchy sequence in $\mathscr{C}(X)$ is convergent.

Let $\{f_n\}$ be a Cauchy sequence in $\mathscr{C}(X)$.

$\Rightarrow \forall \epsilon > 0$, corresponds an N such that $\|f_n - f_m\| < \epsilon$ if $n \geq N$, $m \geq N$.

By known theorem (7.8), there is a function with domain to which $\{f_n\}$ converges uniformly.

By theorem ~~7.12~~ 5 [Book 7.12], f is continuous.

Also, f is bounded (since there is an n such that $|f(x) - f_n(x)| < 1$ $\forall x$)

$\therefore f \in \mathscr{C}(X)$.

Since $f_n \to f$ uniformly on X, we have $\|f_n - f\| \to 0$ as $n \to \infty$.

$\therefore \{f_n\}$ converges to f in $\mathscr{C}(X)$.

Hence $\mathscr{C}(X)$ is complete.

Uniform convergence and Integration.

Theorem:

Let α be monotonically increasing on $[a,b]$. Suppose $f_n \in \mathscr{R}(\alpha)$ on $[a,b]$ for $n = 1, 2 \ldots$ and suppose $f_n \to f$ uniformly on $[a,b]$. Then $f \in \mathscr{R}(\alpha)$ on $[a,b]$ and

$$\int_a^b f \, d\alpha = \lim_{n \to \infty} \int_a^b f_n \, d\alpha.$$

Proof.

It sufficies to prove this for real f_n.

put $\epsilon_n = \sup |f_n(x) - f(x)| \to$ ①

The supremum being taken over $a \le x \le b$.

① $\Rightarrow$ $|f(x) - f_n(x)| < \epsilon_n$.

$\Rightarrow -\epsilon_n \le f(x) - f_n(x) < \epsilon_n$.

$\Rightarrow -\epsilon_n \le f - f_n < \epsilon_n$

$\Rightarrow f_n - \epsilon_n \le f \le f_n + \epsilon_n$

For any partition P of $[a,b]$

$f \le f_n + \epsilon_n$

$\Rightarrow U(P, f, \alpha) \le U(P, f_n+\epsilon_n, \alpha)$

$\Rightarrow \inf U(P, f, \alpha) \le \inf U(P, f_n+\epsilon_n, \alpha)$

$\Rightarrow \overline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} (f_n+\epsilon_n)\,d\alpha$

$\le \int_a^b (f_n+\epsilon_n)\,d\alpha \quad [\because f_n \in R(\alpha)]$

$\therefore \overline{\int_a^b} f\,d\alpha \le \int_a^b (f_n+\epsilon_n)\,d\alpha \rightarrow (2)$

similarly, $\int_a^b (f_n-\epsilon_n)\,d\alpha \le \underline{\int_a^b} f\,d\alpha \rightarrow (3)$

(3) & (2) $\Rightarrow \int_a^b (f_n-\epsilon_n)\,d\alpha \le \underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha$

$\le \int_a^b (f_n+\epsilon_n)\,d\alpha \rightarrow (4)$

$\Rightarrow \int_a^b f_n\,d\alpha - \int_a^b \epsilon_n\,d\alpha \le \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha$

$\le \int_a^b f_n\,d\alpha + \int_a^b \epsilon_n\,d\alpha.$

$\Rightarrow 0 \le \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha \le \cancel{\int_a^b f_n\,d\alpha} + \int_a^b \epsilon_n\,d\alpha$

$- \cancel{\int_a^b f_n\,d\alpha} + \int_a^b \epsilon_n\,d\alpha.$

$\le 2\epsilon_n \int_a^b d\alpha$

$$0 \le \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha \le 2\epsilon_n\,[\alpha(b) - \alpha(a)] \rightarrow (5)$$

Since $\epsilon_n \to 0$ as $n \to \infty$.

$$(5) \Rightarrow \quad 0 \le \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha \le 0 \rightarrow (6)$$

$$\Rightarrow \overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha.$$

Hence $f \in R(\alpha)$.

Again $(5) \Rightarrow \int_a^b (f_n - \epsilon_n)\,d\alpha \le \underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha$

$$\le \int_a^b (f_n + \epsilon_n)\,d\alpha.$$

Since $f \in R(\alpha)$

$$\int_a^b (f_n - \epsilon_n)\,d\alpha \le \int_a^b f\,d\alpha \le \int_a^b (f_n + \epsilon_n)\,d\alpha$$

$$\Rightarrow \int_a^b f\,d\alpha \le \int_a^b (f_n + \epsilon_n)\,d\alpha$$

$$\int_a^b f\,d\alpha \le \int_a^b f_n\,d\alpha + \epsilon_n \int_a^b d\alpha$$

$$\cancel{\int_a^b f\,d\alpha \le \int_a^b f_n\,d\alpha +}$$

$$\int_a^b f\,d\alpha - \int_a^b f_n\,d\alpha \le \epsilon_n(\alpha(b) - \alpha(a)),$$

$$\Rightarrow 0 \le \left| \int_a^b f\,d\alpha - \int_a^b f_n\,d\alpha \right| \le \epsilon_n(\alpha(b)-\alpha(a)).$$

As $n\to\infty$, $\epsilon_n \to 0$.

$$\Rightarrow 0 \le \left| \int_a^b f\,d\alpha - \int_a^b f_n\,d\alpha \right| \le 0.$$

$$\Rightarrow \lim_{n\to\infty} \int_a^b f_n\,d\alpha = \int_a^b f\,d\alpha.$$

Corollary:

If $f_n \in \mathcal{R}(\alpha)$ on $[a,b]$ and if

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (a \le x \le b).$$

The series converges uniformly on $[a,b]$.

Then $\int_a^b f\,d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n\,d\alpha$.

(i.e the series may integrated term by term).

Uniform convergence and Differenciation.

Theorem 1:

Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a,b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$ then $\{f_n\}$ converges uniformly on $[a,b]$ to a function f and

$$f'(x) = \lim_{n\to\infty} f_n'(x) \quad (a \le x \le b).$$

Proof :

Let $\epsilon > 0$ be given.

Choose N such that $n \geq N$, $m \geq N$.

implies $|f_n(x_0) - f_m(x_0)| < \epsilon/2 \rightarrow (1)$

and $|f_n'(t) - f_m'(t)| < \dfrac{\epsilon}{2(b-a)}$ $(a \leq t \leq b)$ $\rightarrow (2)$

If we apply the mean value theorem to the function $f_n - f_m$.

$|f_n(x) - f_m(x) - f_n(t) + f_m(t)| \leq |x-t| \, |f_n'(t) - f_m'(t)|$

$\leq \dfrac{|x-t| \, \epsilon}{2(b-a)}$ (by (2))

$\leq \epsilon/2 \rightarrow (3)$.

Now,

for any x & t on $[a,b]$ if $n \geq N$, $m \geq N$.

$|f_n(x) - f_m(x)| = |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0) + f_n(x_0) - f_m(x_0)|$

$\leq |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)|$.

$\leq \epsilon/2 + \epsilon/2$

$\Rightarrow |f_n(x) - f_m(x)| < \epsilon$.

$\Rightarrow \{f_n\}$ converges uniformly on $[a,b]$

Let $f(x) = \lim_{n\to\infty} f_n(x) \quad (a \le x \le b)$

Now, fix a point x on $[a,b]$ and define

$$\left.\begin{aligned} \phi_n(t) &= \frac{f_n(t) - f_n(x)}{t-x} \\ \phi(t) &= \frac{f(t) - f(x)}{t-x} \end{aligned}\right\} \rightarrow (4).$$

for $a \le t \le b$, $t \ne x$

Then $\lim_{t\to x} \phi_n(t) = \lim_{t\to x} \frac{f_n(t) - f_n(x)}{t-x}$

$$= f_n'(x) \rightarrow (5)$$

$$(3) \Rightarrow \left| \frac{f_n(x) - f_n(t)}{t-x} - \frac{f_m(x) - f_m(t)}{t-x} \right| < \epsilon. \quad n = 1, 2 \ldots$$

$$\Rightarrow |\phi_n(t) - \phi_m(t)| < \epsilon \quad \forall\, n, m \ge N.$$

$\therefore \{\phi_n\}$ converges uniformly for $t \ne x$. Since $\{f_n\}$ converges to f,

$$(4) \Rightarrow \lim_{n\to\infty} \phi_n(t) = \phi(t) \rightarrow (6)$$

uniformly for $a \le t \le b$, $t \ne x$.

By known theorem 4, (7.11),

(5) & (6) $\Rightarrow \lim_{t \to x} \phi(t) = \lim_{n \to \infty} f_n'(x)$

(i.e) $\lim_{t \to x} \frac{f(t) - f(x)}{t - x} = \lim_{n \to \infty} f_n'(x)$ (by (4))

$\Rightarrow f'(x) = \lim_{n \to \infty} f_n'(x).$