UNIT - V

Q.7. a) $A \in L(R^n, R^m)$ then $\|A\| < \infty$.

Let $\{e_1, \dots e_n\}$ be the standard basis in R^n and suppose $x = \sum c_i e_i$, $|x| \leq 1$, so that $|c_i| \leq 1$ for $i = 1, 2 \dots n$.

Then,

$$
\begin{aligned}
|Ax| &= |A \sum c_i e_i| \\
&= |\sum c_i A e_i| \\
&\leq \sum |c_i| \, |Ae_i| \\
&\leq \sum |Ae_i|.
\end{aligned}
$$

So that

$$\|A\| \leq \sum_{i=1}^{n} |Ae_i| < \infty.$$

Since

$$
\begin{aligned}
|Ax - Ay| &= |A(x-y)| \\
&\leq \|A\| \, |x-y|
\end{aligned}
$$

if $x, y \in R^n$, we see that A is uniformly continuous.

b) $\|A+B\| \leq \|A\| + \|B\|$, $\|cA\| = |c| \, \|A\|$.

The inequality in (b) follows from.

$$
\begin{aligned}
|(A+B)x| &= |Ax + Bx| \\
&\leq |Ax| + |Bx| \\
&\leq \|A\| \, |x| + \|B\| \, |x|
\end{aligned}
$$

$\|A+B\| \, |x| \leq (\|A\| + \|B\|) \, |x|$.

$|CAx| \leq |C| \, |Ax|$

$\|CA\| \, |x| \leq |C| \, \|A\| \, |x|$

If $A, B, C \in L(R^n, R^m)$ we have the

triangle inequality.

$$\|A-C\| = \|A-B+B-C\|$$
$$= \|(A-B)+(B-C)\|$$
$$\leq \|A-B\| + \|B-C\|$$

and it is easily verified that $\|A-B\|$ has the other properties of a metric.

c) $\|BA\| \leq \|B\| \, \|A\|$

Finally, (c) follows from.

$$|(BA)x| = |B(Ax)|$$
$$\leq \|B\| \, |Ax|$$
$$\leq \|B\| \, \|A\| \, |x|.$$

Since we now have matrices in the spaces $L(R^n, R^m)$, the concepts of open set, continuity.

9.8.
a) $\|B-A\| \cdot \|A^{-1}\| < 1$. then $B \in \Omega$.

Put $\|A^{-1}\| = \frac{1}{\alpha}$, put $\|B-A\| = \beta$

Given, $\|B-A\| \cdot \|A^{-1}\| < 1$

$$\beta \cdot 1/\alpha < 1.$$
$$\Rightarrow \beta < \alpha.$$

Then $\beta < \alpha$.

For every $x \in R^n$,

$$
\begin{aligned}
\alpha |x| &= \alpha |A^{-1} A x| \\
&\leq \alpha \|A^{-1}\| |Ax| \\
&\leq \alpha \times \frac{1}{\alpha} |Ax| \\
&\leq |Ax| \\
&\leq |Ax - Bx + Bx| \\
&\leq |(A-B)x + Bx| \\
&\leq |(A-B)x| + |Bx| \\
&\leq \beta |x| + |Bx|.
\end{aligned}
$$

So that,

$$(\alpha - \beta)|x| \leq |Bx| \quad \text{—— ①} \quad (x \in R^n).$$

Since $\alpha - \beta > 0$, ① shows that $Bx \neq 0$ if $x \neq 0$.

Hence B is 1-1.

By our known theorem:

" A linear operator A on a finite-dimensional vector space X is one-to-one if and only if the range of A is all of X."

$\therefore B \in \Omega$.

This holds for all B with,

$$\|B-A\| < \frac{1}{\|A^{-1}\|} < \frac{1}{1/\alpha} = \alpha.$$

$$\|B-A\| < \alpha.$$

Thus we have (a) and the fact that

Ω is open.

b) The mapping $A \to A^{-1}$ is continuous.

Replace x by $B^{-1}y$ in ①.

The resulting inequality,

$$(\alpha-\beta)\,|B^{-1}y| \leq |BB^{-1}y|$$
$$\leq |y| \qquad (y \in R^n)$$

Shows that $\|B^{-1}\|\,|y|\,(\alpha-\beta) \leq |y|$.

$$\|B^{-1}\| \leq \frac{1}{\alpha-\beta}$$

$$\|B^{-1}\| \leq (\alpha-\beta)^{-1}$$

The identity,

$$B^{-1}-A^{-1} = B^{-1}(A-B)A^{-1} \qquad \text{— ②}$$

By our known theorem:

" If $A \in L(R^n, R^m)$ and $B \in L(R^m, R^k)$ then $\|BA\| \leq \|B\|\,\|A\|$ "

From ②

$$\|B^{-1}-A^{-1}\| \leq \|B^{-1}\|\,\|A-B\|\,\|A^{-1}\|$$
$$\leq \frac{1}{\alpha-\beta}\cdot\beta\cdot\frac{1}{\alpha}$$
$$\leq \frac{\beta}{\alpha(\alpha-\beta)}$$

$\|B^{-1}-A^{-1}\| < \epsilon$ $\qquad [\because \lim_{t \to x} f(x) = f(t)]$

$\therefore B^{-1} \to A^{-1}$

Then $B \to A$.

9.19 Remarks:

$$\lim_{h \to 0} \frac{|f(x+h)-f(x)-Ah|}{|h|} = 0$$

We know that $f'(x)h = A$.

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - f'(x)h}{h} = 0.$$

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - \frac{f'(x)h}{h} = 0$$

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - f'(x) = 0$$

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

$$\frac{f(x+h) - f(x)}{h} = f'(x)$$

$$f(x+h) - f(x) = hf'(x) + 0.$$

$$f(x+h) - f(x) = hf'(x) + r(h)$$

when $h \to 0$ then $r(h) \to 0$.

9.14 Example:

If $A \in L(R^n, R^m)$ and if $x \in R^n$, then

$$A'(x) = A. \quad \text{—— } (1)$$

Note that x appears on the left side of (1), but not on the right. Both sides of (1) are members of $L(R^n, R^m)$ whereas $Ax \in R^m$.

The proof of (1) is a trivality,

Since

$$f(x+h) - f(x) - Ah = 0.$$

$$f(x+h) - f(x) = Ah \quad [f = A]$$

$$A(x+h) - A(x) = Ah.$$

by the linearity of A.

With $f(x) = A(x)$

$$\lim_{h\to 0} \frac{|f(x+h) - f(x) - Ah|}{|h|} = 0.$$

$$\lim_{h\to 0} \frac{|A(x+h) - A(x) - Ah|}{|h|} = 0$$

$$\lim_{h\to 0} \frac{|A(x) + Ah - A(x) - Ah|}{|h|} = 0.$$

The numerator is zero for every $h \in R^n$.

then the equations becomes,

$$f(x+h) - f(x) = f'(x)h + r(h)$$

$$f(x+h) - f(x) - f'(x)h = r(h) \qquad [f'(x) = A]$$

$$f(x+h) - f(x) - Ah = rh.$$

$$0 = rh.$$

$$\therefore \boxed{rh = 0}$$

9.15. Chain rule:

$F(x) = g(f(x))$ ⓢ $F'(x_0) = g'(f(x_0)) f'(x_0)$

Proof:

Put $y_0 = f(x_0)$

$A = f'(x_0)$

$B = g'[f(x_0)] = g'[y_0] \qquad [\because y_0 = f(x_0)]$

We know that,

$$f(x+h) - f(x) - Ah \in R^m$$

From this we can write,

$$u(h) = f(x_0+h) - f(x_0) - Ah. \quad \text{—— (1)}$$

$$v(k) = g(y_0+k) - g(y_0) - Bk \quad \text{—— (2)}$$

for all $h \in R^n$ and $k \in R^m$ for which $f(x_0+h)$ and $g(y_0+k)$ are defined.

then,

w.k. that the condition,

$$\lim_{h\to 0} \frac{r(h)}{h} = 0.$$

From this we can write as follows:

$$\left.\begin{array}{l} \dfrac{|u(h)|}{|h|} = \epsilon(h) \text{ and } \dfrac{|v(k)|}{|k|} = \eta(k) \\ \Rightarrow |u(h)| = \epsilon(h)|h| \text{ and } |v(k)| = \eta(k)|k| \end{array}\right\} \quad (3)$$

where $\epsilon(h) \to 0$ as $h \to 0$ and $\eta(k) \to 0$ as $k \to 0$.

Given h, put $k = f(x_0+h) - f(x_0)$. — (4)

Then,

$$(1) \Rightarrow u(h) = f(x_0+h) - f(x_0) - Ah.$$

$$u(h) + Ah = f(x_0+h) - f(x_0)$$

$$k = u(h) + Ah \qquad (\text{by } (4))$$

$$|k| = |u(h) + Ah|$$

$$\leq |u(h)| + |Ah| \leq \|A\| |h| + \epsilon(h)|h|$$

$$\leq |h| [\|A\| + \epsilon(h)] \qquad [\text{by } (3)]$$

$$|k| \leq [\|A\| + \epsilon(h)] |h| \quad \text{———} \quad (5)$$

Then we get,

$$F(x_0+h) - F(x_0) - BAh = g[y_0+k] - g[y_0] - BAh.$$

$$= g[y_0+k] - g[y_0] - BAh + Bk - Bk \qquad [+(1) - Bk]$$

$$= \{g[y_0+k] - g[y_0] - Bk\} - BAh + Bk.$$

$$= v(k) + B[k - Ah] \qquad [\text{by } (3)]$$

$$= v(k) + B[f(x_0+h) - f(x_0) - Ah] \qquad [\text{by } (4)]$$

$$F(x_0+h) - F(x_0) - BAh = Bu(h) + v(k) \qquad [\text{by } (1)]$$

$$|F(x_0+h) - F(x_0) - BAh| = |Bu(h) + v(k)|.$$

$$\leq |Bu(h)| + |v(k)|$$

$\leq \|B\| \, |u(h)| + \eta(k) \, |k|$ [by (3)]

$\leq \|B\| \, |h| \, \epsilon(h) + \eta(k) [\, \|A\| + \epsilon(h)] \, |h|$ [by (5)]

$< |h| \, [\, \|B\| \, \epsilon(h) + \eta(k) [\, \|A\| + \epsilon(h)]]$

$$\frac{|F(x_0+h) - F(x_0) - BAh|}{|h|} \leq \|B\| \, \epsilon(h) + \eta(k) [\, \|A\| + \epsilon(h)]$$

Let $h \to 0$. then $\epsilon(h) \to 0$. Also $k \to 0$ so that $\eta(k) \to 0$.

$$\lim_{h \to 0} \frac{|F(x_0+h) - F(x_0) - BAh|}{|h|} = 0$$

$$f'(x_0) = 0.$$

$$f(x) = g[f(x)]$$

$$f'(x) = g'[f(x)] \, f'(x)$$

$$f'(x_0) = g'[f(x_0)] \, f'(x_0)$$

$$= g'(y_0) \, f'(x_0)$$

$$f'(x_0) = BA.$$

9.19. $\|f'(x)\| \leq M$ then $|f(b) - f(a)| \leq M|b-a|$.

Proof:

Given: $\|f'(x)\| \leq M$ — (1)

For $a \in E$, $b \in E$. Define

$\gamma(t) = (1-t)a + tb$. — (2)

for all $t \in R^1$. Such that $\gamma(t) \in E$.

Since E is convex,

$\gamma(t) \in E$ if $0 \leq t \leq 1$.

Put, $g(t) = f(\gamma(t))$

Then by chain rule theorem,

$$g'(t) = f'(\gamma(t))\gamma'(t)$$

Differentiating (2) we get,

$$\gamma'(t) = b-a.$$

$$g'(t) = f'[\gamma(t)]\,(b-a)$$

so that,

$$|g'(t)| = |f'[\gamma(t)]\,\gamma'(t)|$$
$$= |f'[\gamma(t)]\,(b-a)|$$
$$\leq \|f'[\gamma(t)]\|\,|b-a|.$$

$$|g'(t)| \leq M\,|b-a| \quad \text{[by (1)]} \qquad \|f'(x)\| \leq M.$$

for all $t \in [0,1]$.

By our known theorem:

" Suppose f is a continuous mapping of $[a,b]$ into R^k and f is differentiable in $[a,b]$ then there exist $x \in (a,b)$ such that,

$$|f(b)-f(a)| \leq (b-a)\,|f'(x)|"$$

$$|f(b)-f(a)| \leq (b-a)\,|f'(x)|. \quad \text{— (A)}$$

Sub $a=0$, $b=1$ in (A) $\qquad [g(0)=f(a)$, $g(1)=f(b)]$

$$|g(1)-g(0)| \leq M\,|b-a|$$

But $g(0)=f(a)$, $g(1)=f(b)$

$$\therefore |f(b)-f(a)| \leq M\,|b-a|.$$

9.21. $f \in \mathscr{C}'(E) \Leftrightarrow$ partial derivatives $D_j f_i$ exists.

Proof:

Assume first that $f \in \mathscr{C}'(E)$.

We have to prove that $D_j f_i$ exist and continuous.

By our known result,

$$f'(x) e_j = \sum_{i=1}^{m} (D_j f_i)(x) u_i \quad (1 \leq j \leq n) \quad \text{——} \; (1)$$

From (1),

$$[f'(x) e_j] \cdot u_i = (D_j f_i)(x) \quad \text{——} \; (2)$$

$$(D_j f_i)(x) = [f'(x) e_j] \cdot u_i$$

for all i, j and for all $x \in E$. Hence,

$$(D_j f_i)(y) = (f'(y) e_j) \cdot u_i \quad \text{——} \; (3)$$

(3) $-$ (2)

$$\Rightarrow (D_j f_i)(y) - (D_j f_i)(x) = \{(f'(y) e_j) u_i - (f'(x) e_j) \cdot u_i\}$$

$$= \{[f'(y) - f'(x)] e_j\} \cdot u_i.$$

and since $|u_i| = |e_j| = 1$, it follows that

$$|(D_j f_i)(y) - (D_j f_i)(x)| = |\{[f'(y) - f'(x)] e_j\} \cdot u_i|$$

$$\leq |[f'(y) - f'(x)] e_j| \, |u_i|$$

$$\leq |[f'(y) - f'(x)] e_j|$$

$$\leq \|f'(y) - f'(x)\| \, |e_j|$$

$$\leq \|f'(y) - f'(x)\|$$

$$\leq \epsilon$$

Hence $D_j f_i$ is continuous.

For the converse, it suffices to consider the case $m=1$.

Fix $x \in E$ and $\epsilon > 0$.

Since E is open, there is an open ball $S \subset E$, with center at x and radius r and the continuity of the functions $D_j f$ shows that r can be chosen so that,

$$|(D_j f)(y) - (D_j f)(x)| < \epsilon/n \quad (y \in S,\ 1 \le j \le n). \text{ — (A)}$$

Suppose $h = \sum h_j e_j \cdot |h| < r$, put $v_0 = 0$ and $v_k = h_1 e_1 + h_2 e_2 + \cdots + h_k e_k$ for $1 \le k \le n$.

Then,

$$f(x+h) - f(x) = \sum_{j=1}^{n} [f(x+v_j) - f(x+v_{j-1})] \text{ — (4)}$$

Since $|v_k| < r$ for $1 \le k \le n$ and since S is convex, the segments with end points $x+v_{j-1}$ and $x+v_j$ lie in S.

Since $v_j = v_{j-1} + h_j e_j$

The mean value theorem.

"If f is a real continuous function on $[a,b]$ which is differentiable in (a,b), then there is a point $x \in (a,b)$ at which

$f(b) - f(a) = (b-a) f'(x)$."

By this theorem, the jth summand in (4) is equal to,

$h_j (D_j f)(x + v_{j-1} + \theta_j h_j e_j)$

for some $\theta_j \in (0,1)$, and this differs from $h_j(D_jf)(x)$ by less than $|h_j| \epsilon/n$. using (A).

~~using~~ By (4), it follows that.

$$|f(x+h) - f(x) - \sum_{j=1}^{n} h_j (D_jf)(x)|$$

$$\leq \frac{1}{n} \sum_{j=1}^{n} |h_j| \epsilon$$

$$\leq |h| \epsilon.$$

for all h such that $|h| < r$.

This says that f is differentiable at x and that $f'(x)$ is the linear function which assigns the number $\sum h_j (D_jf)(x)$ to the vector

$$h = \sum h_j e_j.$$

The matrix $[f'(x)]$ consists of the row $(D_1f)(x) \cdots (D_nf)(x)$; and since $D_1f \cdots D_nf$ are continuous functions on E.

$$\therefore f \in \mathscr{C}'(E)$$

Hence Proved.