Properties:-

$$\int_a^b f(x)\,dx = -\int_a^b F(x)\,dx$$

Proof:-

LHS :- $\int_a^b F(x)\,dx = [F(x)]_a^b$

$= F(b) - F(a) \rightarrow (1)$

RHS $= -\int_b^a f(x)\,dx = -[F(x)]_b^a$

$= -[F(a) - F(b)] \Rightarrow F(b) - F(a) \rightarrow (2)$

From (1) and (2) we get.

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Eg: $\int_1^2 x^2\,dx = -\int_2^1 x^2\,dx = 7/3$

Property:03

$\int_a^b F(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where c is a between a and b.

proof:-

LHS : $\int_a^b f(x)\,dx = [F(x)]_a^b$

$= F(b) - F(a) \rightarrow (1)$

RHS: $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$= [F(x)]_a^c + [F(x)]_c^b$

$= [F(c) - F(a)] + [F(b) - F(c)]$

$= F(b) - F(a) \rightarrow (2)$

From ① and ② we get.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Eg: $\int_1^3 (x+x^2)\,dx = \int_1^2 (x+x^2)\,dx + \int_2^3 (x+x^2)\,dx$

where 2 lies between ① and ③

Property :02 :-

$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy.$$

Proof:-

LHS :- $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b)-F(a) \rightarrow ①$

RHS :- $\int_a^b f(y)\,dy = [F(y)]_a^b = F(b)-F(a) \rightarrow ②$

From ① and ②

$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy \Rightarrow \int_1^3 x^3\,dx = \int_1^2 y^3\,dy =$

Property :04 :-

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

Proof:-

RHS $= \int_0^a f(a-x)\,dx$

put $a-x=y \Rightarrow -dx = dy$ when $x=0$; $y=a$

$x=a$, $y=0$

① $\Rightarrow \int_0^a f(a-x)\,dx = \int_a^0 f(y)(-dy)$

$= -\int_a^0 f(y)\,dy \Rightarrow \int_0^a f(y)\,dy$ (property(1))

$= \int_0^a f(x)\,dx$

Hence

$$\int_0^a f(x)\,dx = \int_0^a \cancel{f(y)d}\ f(a-x)\,dx.$$

Property 5:-

$$\int_{-a}^a f(x)\,dx = \int_{-a}^a f(x)\,dx + \int_a^0 f(x)\,dx$$

Proof:-

$$\int_{-a}^a f(x)\,dx = \int_{-a}^a f(x)\,dx + \int_a^0 f(x)\,dx.$$

In the first integral on the right, put $x=-y$ when $x=-a$, $y=0$, $x=0$, $y=0$.

$$\int_{-a}^0 f(x)\,dx = \int_a^0 f(-y)(-dy)$$

$= -\int_a^0 f(-y)\,dy$

$= \int_0^a f(-y)\,dy$ [By property ①]

$= \int_0^a f(x)\,dx$ · [By property 2] $\rightarrow$ ②

substituting ② in ① we get

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

corollory, If $f(x)$ is an odd function

(ie) $f(-x) = -f(x)$ then

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

[In the above result by replacing $f(-x)$ by $-f(x)$]

$$\therefore \int_{0}^{a} f(x)\,dx = 0 \quad \text{[when } f(x) \text{ is odd]}$$

Eg :- $\int_{-\pi/2}^{\pi/2} \sin^5 x\,dx = 0$ [$\sin^5 x$ is an odd function]

$\int_{-\pi/2}^{\pi/2} \sin^3 x\,dx = 0$ [$\because$ odd function].

collary 2 :-

If $f(x)$ is an even function.

$f(-x) - f(x)$ When $\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$

[In the above result by replacing

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

Eg: 1 $\int_{-\pi/2}^{\pi/2} \cos^4 dx = 2\int_0^{\pi/2} \cos^4 x\, dx.$

[$\cos^4 x$ is even func]

Eg: 2 $\int_{-1}^{1} x^6 dx = 2\int_0^1 x^6 dx$ [x^6 is an even func]

Eg: 3 $\int_{-\pi/2}^{\pi/2} \sin^4 x\, dx = 2\int_0^{\pi/2} \sin^4 x\, dx$

[$\sin^4 x$ is even func]

$\because \sin(-x) = -\sin x.$

$\sin^4(-x) = (-\sin^4 x) = \sin^4 x,$

Property: 6:-

$$\int_0^{2a} f(x)dx = \int_0^a f(x)\, dx + \int_a^{2a} f(x)\, dx$$

Proof:-

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx$$

In the second integral on the right.

put $x = 2a - y$

$dx = -dy.$

When $x = a;\ y = 0$

$x = 2a;\ y = 0$

$$\int_a^{2a} f(x)\, dx = \int_a^0 f(2a-y)(-dy)$$

$$= -\int_a^0 f(2a-y)\, dy$$

$$= \int_0^a f(2a-y)\,dy \quad [\text{by property } ①]$$

$$= \int_0^a f(2a-x)\,dx \quad [\text{by property } ②]$$

$$\int_0^{2a} f(x)\,dx = \int_0^a f(2a-x)\,dx \rightarrow ②$$

Sub ② in ①

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx.$$

corollary:-

If $f(x) = f(2a-x)$ then the above integral becomes $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$

proof:-

$$\int_0^{\pi} \sin^n x\,dx = \int_0^{2(\pi/2)} \sin^n x\,dx = 2\int_0^{\pi/2} \sin^n x\,dx$$

$\because \sin^n(\pi - x) = \sin^n x$

Example 5:-

Evaluate $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$

Soln:-

Let $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx \rightarrow (1)$

By using the property $\int_0^a f(x) dx = \int_0^a f(a-x) dx$

we get

$$I = \int_0^{\pi/2} \frac{\sin^2(\pi/2 - x)}{\sin(\frac{\pi}{2} - x) + \cos(\frac{\pi}{2} - x)} dx$$

$$= \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x} dx \rightarrow (2)$$

Adding (1) and (2)

$$2I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx + \int_0^{\pi/2} \frac{\cos^2 x}{\sin x + \cos x} dx$$

$$= \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$$

put $t = \tan x/2$.

$$dx = \frac{2dt}{1+t^2}\ ; \ \sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

when $x=0$, $t=0$

when $x=\pi/2$; $t=1$

$$2I = \int_0^1 \frac{2dt/1+t^2}{\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}$$

$$= 2\int_0^1 \frac{dt}{-t^2+2t+1} \Rightarrow 2\int_0^1 \frac{dt}{-(t-1)^2+2}$$

$$= 2\int_0^1 \frac{dt}{(\sqrt{2})^2-(t-1)^2}$$

$$= 2\left[\frac{1}{2\sqrt{2}}\log\frac{\sqrt{2}+(t-1)}{\sqrt{2}-(t-1)}\right]_0^1$$

$$= \frac{1}{\sqrt{2}}\left[\log 1 - \log\frac{\sqrt{2}-1}{\sqrt{2}+1}\right]$$

$$= \frac{1}{\sqrt{2}}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$= \frac{1}{\sqrt{2}}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}$$

$$2I = \frac{1}{\sqrt{2}}\log\frac{(\sqrt{2}+1)^2}{2-1} \Rightarrow \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)^2$$

$$2I = \frac{2}{\sqrt{2}}\log(\sqrt{2}+1)$$

$$\boxed{I = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)}$$

Example : 6 :-

Evaluate $\int_0^{\pi/2} \log(\sin x)\, dx$.

Sol :-

$$I = \int_0^{\pi/2} \log(\sin x)\, dx \rightarrow ①$$

$$I = \int_0^{\pi/2} \log(\sin(\pi/2 - x))\, dx$$ [∵ using property 3]

$$I = \int_0^{\pi/2} \log \cos x\, dx \rightarrow ②$$

Add ① + ②

$$2I = \int_0^{\pi/2} \log(\sin x)\, dx + \log \cos x\, dx$$

$$= \int_0^{\pi/2} \log(\sin x \cos x)\, dx.$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx.$$ ∵ $\sin 2x = 2\sin x \cos x$

$$= \int_0^{\pi/2} (\log \sin 2x - \log 2)\, dx$$

$$= \int_0^{\pi/2} \log \sin 2x\, dx - \int_0^{\pi/2} \log 2\, dx$$

$$2I = \int_0^{\pi/2} \log \sin 2x\, dx - \pi/2 \log 2 \rightarrow ③$$

To evaluate $\int_0^{\pi/2} \log(\sin 2x)\, dx$

Put $2x = y$

Differentiate

$\frac{dy}{dx} = 2$

$\boxed{dy = 2.dx}$ when $2x, y=0$; $x = \pi/2$ $y = 0$

$= \int_0^{\pi/2} \log(\sin 2x)\, dx \Rightarrow \int_0^{\pi/2} \log(\sin y)\, dy.$

$= \frac{1}{2} \cdot 2 \int_0^{\pi/2} \log(\sin y)\, dy \Rightarrow \int_0^{\pi/2} \log(\sin y)\, dy$

$= \int_0^{\pi/2} \log \sin x \, dx$

$= I$ $\left[\because \int_0^a f(x)\, dx = \int_0^a f(y)\, dy \right]$

i.e $= \int_0^{\pi/2} \log(\sin 2x)\, dx = I \rightarrow (4)$

Sub (4) in (2)

$2I = I - \pi/2 \log 2$

$I = -\pi/2 \log 2$

$\int_0^{\pi/2} \log(\sin 2x)\, dx = -\pi/2 \log 2$

$= \pi/2 \log (2^{-1})$

$= \pi/2 \log (1/2)$

Example 7 :-

Evaluate $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$.

Let $I = \int_0^{\pi/2} \frac{x \tan x}{\sec x + \tan x} dx \rightarrow ①$

Also the same.

$$I = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)}$$

$$= \int_0^{\pi} \frac{(\pi - x)(-\tan x)}{-\sec x - \tan x} \qquad \because \tan(\pi - x) = -\tan x$$

$$= \int_0^{\pi} \frac{(\pi - x)\tan x}{\tan x + \sec x} dx \rightarrow ②$$

Add ① and ②

$$2I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx + \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \tan x} dx$$

$$= \int_0^{\pi} \frac{x \tan x + (\pi - x)\tan x}{\sec x + \tan x} dx$$

$$= \pi \int_0^{\pi} \frac{\tan x}{\sec x + \tan x} dx - \pi \int_0^{\pi} \frac{\sin x}{1 + \sin x} dx$$

$$= \pi \int_0^{\pi} \frac{\sin x(1 - \sin x)}{(1 + \sin x)(1 - \sin x)} dx$$

$$= \pi \int_0^{\pi} \frac{\sin x - \sin^2 x}{1 - \sin^2 x} dx$$

$$= \pi \int_0^{\pi} \frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} dx$$

$$= \pi \int_0^{\pi} (\sec x \tan x - \tan^2 x) dx$$

$$= \{\pi [(\sec x]_0^{\pi} + (\tan x)_0^{\pi} + (x)_0^{\pi}]$$

$$= \pi (\sec \pi - \sec 0 + \tan \pi - \tan 0 + \pi - 0)$$

$$2I = \pi [-1 - 1 + \pi] \Rightarrow \pi(\pi - 2)$$

$$I = \pi/2 (\pi - 2)$$

P.T $\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx$.

Let $f(x) = \sin^n x$; Here $a = \pi/2$

$$f(a - x) = \sin^n (\pi/2 - x)$$

$$= \cos^n x$$

from property $\left[\int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$