12.09.20

1) $y = \dfrac{x+1}{x+2}$ find $\dfrac{d^2y}{dx^2}$.

$y = (x+1)(x+2)^{-1}$

$y' = 1\ (-1)(x+2)^{-2}$

$d\left(\frac{u}{v}\right) = \dfrac{du.v - u.dv}{v^2}$

$\dfrac{dy}{dx} = \dfrac{(x+2)(1) - (x+1)(1)}{(x+2)^2}$

$y' = -1(x+2)^{-2}$

$y' = -1(-2)(x+2)^{-3}$

$= \dfrac{x+2-x-1}{(x+2)^2} = \dfrac{1}{(x+2)^2}$ $\quad \dfrac{2}{(x+2)^3}$

$\dfrac{d^2y}{dx^2} = (x+2)^{-2}$ $\quad d(x^n) = n x^{n-1}$

$= -2(x+2)^{-2-1}$

$= -2(x+2)^{-3}$ (or) $\dfrac{-2}{(x+2)^3}$

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2) If $y = a\cos 2x + b\sin 3x$ find $\dfrac{d^2y}{dx^2}$

$y' = a(-\sin 2x)(2) + b\cos 3x\ (3)$

$y' = -2a\sin 2x + 3b\cos 3x$

$y'' = -2a\cos 2x(2) + 3b(-\sin 3x)(3)$

$y'' = -4a\cos 2x - 9b\sin 3x.$

$y'' = -[4a\cos 2x + 9b\sin 3x]$

3) If $y = ae^{mx} + be^{-mx}$ prove that $y'' - m^2 y = 0$

$$y' = ae^{mx}(m) + be^{-mx}(-m)$$

$$y'' = ae^{mx}m^2 + be^{-mx}m^2$$

$$y'' = m^2[ae^{mx} + be^{-mx}]$$

$$y'' = m^2 y$$

$$y'' - m^2 y = 0.$$

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4) $y = a\cos(\log x) + b\sin(\log x)$ prove that

$x^2y'' + xy' + y = 0$

$$y' = a - \sin(\log x)\frac{1}{x} + b\cos(\log x)\frac{1}{x}.$$

$$xy' = -a\sin\log x + b\cos(\log x)$$

$$xy'' + y' = -a\cos(\log x)\frac{1}{x} + b\left(-\sin(\log x)\cdot\frac{1}{x}\right)$$

$$x^2y'' + xy' = -a\cos\log x - b\sin\log x.$$

$$x^2y'' + xy' = -y$$

$$x^2y'' + xy' + y = 0.$$

Hence proved.

(5) $x = a(\cos\theta + \theta\sin\theta)$; $y = a(\sin\theta - \theta\cos\theta)$

find y''

$$\frac{dx}{d\theta} = a(\theta\cos\theta + \sin\theta(1) - \sin\theta)$$

$$\frac{dx}{d\theta} = a\theta\cos\theta \rightarrow (1)$$

$$\frac{dy}{d\theta} = a[\cos\theta - (\theta(-\sin\theta) + \cos\theta(1))]$$

$$= a[\cos\theta + \theta\sin\theta - \cos\theta]$$

$$dy/d\theta = a\theta\sin\theta \rightarrow (2)$$

$$\frac{dy/d\theta}{dx/d\theta} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

$$\frac{dy}{dx} = \tan\theta$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan\theta)$$

$$= \frac{d(\tan\theta)}{dx} \times \frac{d\theta}{d\theta}$$

$$= \frac{d(\tan\theta)}{d\theta} \times \frac{d\theta}{dx}$$

$$= \sec^2\theta \cdot \frac{1}{a\theta\cos\theta}$$

$$y'' = \frac{\sec^3\theta}{a\theta}$$

6) $y = (\sin^{-1} x)^2$ prove that $(1-x^2)y_2 - xy_1 = 2$

Given

$$y = (\sin^{-1} x)^2$$

$$y' = 2\sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}}$$

$$y'\sqrt{1-x^2} = 2\sin^{-1} x$$

Squaring both sides

$$(y')^2(1-x^2) = 4(\sin^{-1} x)^2 \qquad y = (\sin^{-1} x)^2$$

Differentiating on both sides with respect 'x'

we get

$$2y'y''(1-x^2) + (y')^2(-2x) = 4y'$$

$$2y'y''(1-x^2) + (y')^2(-2x) - 4y' = 0$$

$$2y'(y''(1-x^2) + y'(-x) - 2) = 0$$

$$(1-x^2)y'' - xy' - 2 = 0$$

$$(1-x^2)y'' - xy' = 2$$

Hence it's proved.

7) $y = e^{\tan^{-1} x}$ prove that $(1+x^2)y'' + (2x-1)y' = 0$.

Given $y = e^{\tan^{-1} x}$

$$y' = e^{\tan^{-1} x} \cdot \frac{1}{1+x^2}$$

$$(1+x^2)\, y' = e^{\tan^{-1} x}.$$

Differentiating again on both sides with respect 'x' we get

$$(1+x^2)\, y'' + (2x) y' = y'$$

$$(1+x^2)\, y'' + 2xy' - y' = 0$$

$(1+x^2)\, y'' + y'(2x-1) = 0$, Hence's proved,

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8) $y = e^{x} \tan^{-1} x$ prove that $(1+x^2) y_2 - 2(1+x^2-x) y_1 + (1-x^2) y = 0$.

$$y = e^{x} \tan^{-1} x$$

$$y' = e^{x} \tan^{-1} x + e^{x} \frac{1}{1+x^2}$$

$(1+x^2)$ multiply on both sides.

$$(1+x^2)\, y' = e^{x} \tan^{-1} x \,(1+x^2) + e^{x}.$$

$(1+x^2)y_1 - (1+x^2)y = e^x \rightarrow ①$

Differentiating on both sides with respect 'x'

we get.

$2xy' + (1+x^2)y'' - (2xy + (1+x^2)y') = e^x$

$2xy' + (1+x^2)y'' - 2xy - (1+x^2)y' = e^x$

$(1+x^2)y'' + y'(2x - 1 + x^2) - 2xy = e^x \rightarrow ②$

② in ①

$(1+x^2)y'' + y'(2x-1-x^2) - 2xy = (1+x^2)y_1' - (1+x^2)y$

$(1+x^2)y'' + y'(2x-1-x^2-1-x^2) + (1+x^2)y - 2xy = 0$

$(1+x^2)y'' + y'(-2-2x^2+2x) + (1+x^2)y - 2xy = 0$

$(1+x^2)y'' + y'(-2)(x^2+1-x) + y(1+x^2-2x) = 0$

$(1+x^2)y'' - 2y'(1-x+x^2) + y(1-x)^2 = 0$

Hence proved.

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09) $y = Ae^{-kt}\cos(pt+e)$ prove $\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + n^2y = 0.$

where $n^2 = p^2 + k^2$

Given $y = Ae^{-kt}\cos(pt+e)$

Differentiate both sides with respect

to 't' we get

$\frac{dy}{dt} = A[(e^{-kt}(-k)\cos(pt+e)) A$

$\frac{dy}{dt} = Ae^{-kt}(-k)\cos(pt+e) + Ae^{-kt}(-\sin(pt+e)).p$

$= A[-ke^{-kt}\cos(pt+e) - pe^{-kt}\sin(pt+e)]$

$\frac{dy}{dt} = -Ake^{-kt}\cos(pt+e) - Ape^{-kt}\sin(pt+e)$

$= -ky - Ape^{-kt}\sin(pt+e) \to (1)$

Again Differentiate with respect 't' we get

$\frac{d^2y}{dt^2} = -Ap[e^{-kt}(-k)\sin(pt+e) + e^{-kt}\cos(pt+e).p]$

$-k\, dy/dt$

$= Apke^{-kt}\sin(pt+e) - Ap^2e^{-kt}\cos(pt+e)$

$-k\, dy/dt$

$= -p^2y + kApe^{-kt}\sin(pt+e) - k\frac{dy}{dt}$

$= -p^2y + k(-ky - \frac{dy}{dt}) - k\frac{dy}{dt}$

$= -p^2y - k^2y - k\frac{dy}{dt} - k\frac{dy}{dt}$

$= -y(p^2+k^2) - 2k\frac{dy}{dt} \Rightarrow -n^2y - 2k\frac{dy}{dt}$

$\frac{d^2y}{dt^2} + n^2y + 2k\, dy/dt = 0.$ // proved.

(10) $x = (a+bt)e^{-nt}$ prove that $\frac{d^2x}{dt^2} + 2n\frac{dx}{dt} + n^2x = 0$

Given

$$x = (a+bt)e^{-nt}$$

Differentiate on both sides respect 't' we get

$$\frac{dx}{dt} = (a+bt)e^{-nt}(-n) + e^{-nt}(b)$$

$$\frac{dx}{dt} = -ne^{-nt}(a+bt) + be^{-nt}$$

$$\frac{dx}{dt} = -nx + be^{-nt}$$

$$\boxed{\frac{dx}{dt} + nx = be^{-nt}} \rightarrow (1).$$

$$\frac{dx}{dt} = -nx + be^{-nt}$$

differentiate again on both sides

with respect 't' we get

$$\frac{d^2x}{dt^2} = -n\frac{dx}{dt} + be^{-nt}(-n)$$

$$= -n\left[\frac{dx}{dt} + be^{-nt}\right]$$

$$= -n\left[\frac{dx}{dt} + \frac{dx}{dt} + nx\right]$$

$$= -n\left[2\frac{dx}{dt} + nx\right]$$

$$\frac{d^2x}{dt^2} = -2n\frac{dx}{dt} - n^2x$$

$$\frac{d^2x}{dt^2} + n^2x + 2n\frac{dx}{dt} = 0.$$

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(11) $xy = ax^2 + b/x$ prove that $x^2y_2 + 2(xy_1 - y) = 0$

doubt

Given

$xy = ax^2 + b/x \Rightarrow xy - b/x = ax^2$

$x^2y = ax^3 + b$

Differentiating both sides with respect 'x' we get

$2xy + x^2y' = 3ax^2$

$2xy + x^2y' = 3(xy - b/x)$

$(\times x)$ $2x^2y + x^3y' = 3x^2y - 3b$

again differentiating both sides

$4xy + 2x^2y' + 3x^2y' + x^3y'' = 6xy + 3x^2y'$

~~$x^3y'' + 5x^2y' = 2xy$~~

~~$(\div x)$~~

~~$x^2y'' + 5xy' = 2y$~~

$4xy + 2x^2y' + x^3y'' - 6xy = 0$

$x^3y'' + 2x^2y' - 2xy = 0$

$x.[x^2y'' + 2(xy' - y)] = 0$

Hence proved

12) $y = \log\left(\frac{x}{a+bx}\right)^x$ prove that $x^3 y_2 = (y - xy_1)^2$

Given

$$y = \log\left(\frac{x}{a+bx}\right)^x$$

$\log m^n = n \log m$

$\log \frac{m}{n} = \log m - \log n$

$$y = x \log\left(\frac{x}{a+bx}\right)$$

$$y = x(\log x - \log(a+bx))$$

Differentiating on both sides ;

$$y' = x\left[\frac{1}{x} - \frac{1}{a+bx}(b)\right] + \log x - \log(a+bx)$$

$$y_1 = \left[1 - \frac{bx}{a+bx}\right] + \log x - \log(a+bx)$$

$$y_1 = \left[\frac{a + \cancel{bx} - \cancel{bx}}{a+bx}\right] + \log x - \log(a+bx)$$

$$y_1 = \left[\frac{a}{a+bx}\right] + \log\left(\frac{x}{a+bx}\right) \rightarrow (1)$$

$$y = x \log\left(\frac{x}{a+bx}\right)$$

$$y/x = \log\left(\frac{x}{a+bx}\right)$$

$$e^{y/x} = \frac{x}{a+bx} \rightarrow (2)$$

sub ② in ①

$$y_1 = \frac{a}{a+bx} + y/x.$$

Multiply x on both sides

$$xy' = \frac{ax}{a+bx} + y.$$

$$xy' = a\left(\frac{x}{a+bx}\right) + y$$

$$xy' = ae^{y/x} + y \rightarrow ③$$

Differentiating on both sides with respect to 'x' We get

$$xy'' + y' = ae^{y/x}\left[\frac{xy' - y(1)}{x^2}\right] + y'$$

$$xy'' = ae^{y/x}\left[\frac{xy' - y}{x^2}\right] \quad \text{using ⑧}$$

$$xy'' = (xy' - y)\left[\frac{xy' - y}{x^2}\right]$$

$$x^3y'' = (xy' - y)^2 \quad \text{(or)} \quad x^3y'' = (y - xy')^2.$$

Hence proved.