(3) If $y = Ax^{n+1} + Bx^{-n}$ then prove that

$x^2y_2 = n(n+1)y.$

Given $y = Ax^{n+1} + Bx^{-n}$

Differentiate on both sides with respect 'x' we get

$$y' = A(n+1)x^{n+1-1} + B(-n)x^{-n-1}$$

$$y' = A(n+1)x^{n} - nBx^{-(n+1)}$$

Differentiate again on both sides with respect 'x' we get

$$y'' = A(n+1)n\,x^{n-1} - Bn(-n-1)x^{-n-1-1}$$

$$y'' = A\underline{n(n+1)}x^{n-1} + B\underline{n(n+1)}x^{-(n+2)}$$

$$= n(n+1)\left[Ax^{n-1} + Bx^{-(n+2)}\right]$$

Multiply 'x^2' on both sides.

$$x^2y'' = n(n+1)\left[Ax^{n-1}x^2 + Bx^{-(n+2)}x^2\right]$$

$$x^2y'' = n(n+1)\left[Ax^{n+1} + Bx^{-n}\right]$$

$$x^2y'' = n(n+1)y$$

Hence proved.

If $y = ax\cos mx$ proved that $x^2(y_2 + m^2y) = 2(xy_1 - y)$

Given $y = ax\cos mx$

$y' = a\cos mx + ax(-\sin mx)(m)$

$y' = a\cos mx - max\sin mx.$

$y'' = a(-\sin mx)(m) - am\sin mx - max\cos mx(m)$

$= -ma\sin mx - am\sin mx - m^2ax\cos mx$

$y'' = -2am\sin mx - m^2ax\cos mx$

added m^2y on both sides.

$y'' + m^2y = -2am\sin mx - m^2ax\cos mx + m^2ax\cos mx$

$y'' + m^2y = -2am\sin mx$

multiply x^2 on both sides

$x^2(y'' + m^2y) = -2amx^2\sin mx \rightarrow (1)$

$y_1 = a\cos mx - max\sin mx$

$xy_1 - y = ax\cos mx - max^2\sin mx - ax\cos mx$

$xy_1 - y = -max^2\sin mx$

$2(xy_1 - y) = -2max^2\sin mx \rightarrow (2)$

(1) = (2) $\quad x^2(y_2 + m^2y) = 2(xy_1 - y)$

Hence proved.

15). If $y = e^{-2x}\cos 3x$ Find constants 'a' and 'b' such that for all values of $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$

$$y = e^{-2x}\cos 3x$$

$$y_1 = e^{-2x}(-2)\cos 3x + e^{-2x}(-\sin 3x)(3)$$

$$y_1 = -2e^{-2x}\cos 3x - 3\sin 3x\, e^{-2x}$$

$$y_1 = -2e^{-2x}\cos 3x - 3e^{-2x}\sin 3x.$$

$$y_2 = -2(-2)e^{-2x}\cos 3x - 2e^{-2x}(-\sin 3x)(3)$$
$$- 3e^{-2x}(-2)\sin 3x - 3e^{-2x}\cos 3x(3)$$

$$y_2 = 4e^{-2x}\cos 3x + 6e^{-2x}\sin 3x + 6e^{-2x}\sin 3x$$
$$-9e^{-2x}\cos 3x$$

$$y_2 = 12e^{-2x}\sin 3x - 5e^{-2x}\cos 3x$$

$$y_2 = e^{-2x}[12\sin 3x - 5\cos 3x]$$

Given that

$y'' + ay' + by = 0. \Rightarrow e^{-2x}[12\sin 3x - 5\cos 3x] +$

$$a[-2e^{-2x}\cos 3x - 3e^{-2x}\sin 3x] + be^{-2x}\cos 3x = 0$$

$$\Rightarrow 12e^{-2x}\sin 3x - 5e^{-2x}\cos 3x + a(-2)e^{-2x}\cos 3x - 3ae^{-2x}\sin 3x$$
$$+ be^{-2x}\cos 3x = 0$$

$\to 12e^{-2x}\sin 3x - 3ae^{-2x}\sin 3x - 5e^{-2x}\cos 3x - 2ae^{-2x}\cos 3x$

$+ be^{-2x}\cos 3x = 0$

$3e^{-2x}\sin 3x(4-a) + e^{-2x}\cos 3x[-5-2a+b] = 0.$

Equating on both sides with 'a' coefficient

$4-a=0 \to ① \Rightarrow \boxed{a=4}$

$b = 5+2a \to ②$

$a=4$ in ② $\quad b = 5+2a \Rightarrow b = 5+2(4)$

$= 5+8 = 13$

$\boxed{a=4;\ b=13}$

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⑯. If $x = \sin\theta$ and $y = \sin p\theta$ then show that

$(1-x^2)y'' - xy' + p^2y = 0$

Given

$x = \sin\theta \ ; \ y = \sin p\theta$

$\frac{dx}{d\theta} = \cos\theta \ ; \ \frac{dy}{d\theta} = \cos p\theta \cdot p$

$dx = \cos\theta \cdot d\theta \ ; \ dy = \cos p\theta \cdot p \cdot d\theta$

$$\frac{dy}{dx} = \frac{\cos p\theta \cdot p \cdot \cancel{d\theta}}{\cos\theta \cdot \cancel{d\theta}}$$

$$\frac{dy}{dx} = \frac{\cos p\theta \cdot p}{\cos\theta}$$

$d(u \cdot v) = u\,dv + v\,du$

$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{d\theta}\left(\frac{p\cdot\cos p\theta}{\cos\theta}\right)\frac{d\theta}{dx}.$$

$$= \left[\frac{\cos\theta \cdot p^2(-\sin p\theta) - p\cdot\cos p\theta\,(-\sin\theta)}{(\cos\theta)^2}\right]\frac{d\theta}{dx}$$

$$= \left[\frac{-p^2\cos\theta\sin p\theta + p\sin\theta\cos p\theta}{(\cos\theta)^2}\right]\frac{1}{\cos\theta}$$

$$= \left[\frac{-p^2\cancel{\cos\theta}\sin p\theta}{(\cos\theta)^{\cancel{2}}} + \frac{p\sin\theta\cos p\theta}{(\cos\theta)^2}\right]\frac{1}{\cos\theta}$$

$$= \left[\frac{-p^2\sin p\theta}{\cos\theta} + \frac{p\cos p\theta}{\cos\theta}\times\frac{\sin\theta}{\cos\theta}\right]\frac{1}{\cos\theta}$$

$\boxed{\text{m.I}}$

$$= \left[\frac{-p^2 y}{\cos\theta} + \left(\frac{dy}{dx}\cdot\frac{x}{\cos\theta}\right)\right]\frac{1}{\cos\theta}$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{\cos^2\theta}\left[-p^2y + x\frac{dy}{dx}\right]$$

$$\cos^2\theta\left(\frac{d^2y}{dx^2}\right) = -p^2y + x\frac{dy}{dx}.$$

$(1-\sin^2\theta)\dfrac{d^2y}{dx^2} = -p^2y + xy'$

$(1-x^2)y'' - xy' + p^2y = 0$

Hence proved

17). If $x = \log t$, $y = 1/t$ show that $y'' + y' = 0$

Given

$x = \log t$ | $y = 1/t$

$\dfrac{dx}{dt} = \dfrac{1}{t}$ | $\dfrac{dy}{dt} = \dfrac{-1}{t^2}$

$\dfrac{dy}{dx} = \dfrac{-1}{t} \Rightarrow \boxed{y' = -y}$

Differentiate again on both sides

$y'' = -y' \Rightarrow y'' + y' = 0$

18) $\frac{6}{6}$ $y = e^{-x}\cos x$ prove that $\frac{d^4y}{dx^4} + 4y = 0$.

Given $y = e^{-x}\cos x$

$y' = e^{-x}(-1)\cos x + e^{-x}(-\sin x)$

$y' = -e^{-x}\cos x - e^{-x}\sin x$

y_1 or $y' = -y - e^{-x}\sin x \Rightarrow \boxed{e^{-x}\sin x = -y - y'}$

y_2 or $y'' = -y' - \boxed{e^{-x}(-1)}\sin x - e^{-x}\cos x$

$= -y' - y - y' - y \Rightarrow -2y' - 2y$

y_3 or $y''' = -2y'' - 2y' \Rightarrow -2(-2y' - 2y) - 2y'$

$= 4y' + 4y - 2y' \Rightarrow 2y' + 4y$

y_4 or $y'''' = 2y'' + 4y'$

$= 2(-2y' - 2y) + 4y'$

$= \cancel{-4y'} - 4y + \cancel{4y'}$

$y_4 = -4y \Rightarrow y_4 + 4y = 0$

Hence proved

19) If $y = [\log(x+\sqrt{1+x^2})^2]$ then prove that

$(1+x^2)y_2 + xy_1 = 2$

type-1 method: Given $y = \log(x+\sqrt{1+x^2})^2$

Differentiate on both sides with respect 'x'

$$y_1 = 2\log(x+\sqrt{1+x^2}) \cdot \frac{1}{x+\sqrt{1+x^2}} \cdot \left(1+\frac{1}{2}(1+x^2)^{-1/2} \cdot 2x\right)$$

$$= 2\log(x+\sqrt{1+x^2}) \cdot \left\{\frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{x}{\sqrt{1+x^2}}\right)\right\}$$

$$= 2\log(x+\sqrt{1+x^2}) \cdot \left\{\frac{1}{x+\sqrt{1+x^2}}\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)\right\}$$

$$= 2\log(x+\sqrt{1+x^2}) \cdot \frac{1}{\sqrt{1+x^2}}$$

$$y' = \frac{2\log(x+\sqrt{1+x^2})}{\sqrt{1+x^2}} \rightarrow ①$$

Differentiating again on both sides with respect 'x' we get

$$y_2 = 2\left\{\log(x+\sqrt{1+x^2})\frac{-x}{(1+x^2)^{3/2}}\right\} + \frac{1}{\sqrt{1+x^2}}$$

$$\left[1+\frac{x}{\sqrt{1+x^2}}\right]\left[\frac{1}{x+\sqrt{1+x^2}}\right]$$

$$y'' = \frac{-2x \log(x+\sqrt{1+x^2})}{(1+x^2)^{3/2}} + \frac{2}{\sqrt{1+x^2}}\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)\left[\frac{1}{x+\sqrt{1+x^2}}\right]$$

$$y'' = \frac{-2x \log(x+\sqrt{1+x^2})}{(1+x^2)^{3/2}} + \frac{2}{1+x^2}$$

$$= \frac{-xy_1}{1+x^2} + \frac{2}{1+x^2} \quad \text{using (1)}$$

$$y_2 = \frac{2-xy_1}{1+x^2}$$

$$(1+x^2)y_2 + xy_1 = 2$$

Hence proved

20) If $y = \sin(\sin x)$ then prove that

$$y_2 + y_1(\tan x) + y\cos^2 x = 0$$

Given

$$y = \sin(\sin x) \rightarrow (1)$$

$$y' = \cos(\sin x)\cdot\cos x \rightarrow (2)$$

$y_2 = [\cos(\sin x)(-\sin x) + \cos[-\sin(\sin x)\cos x]$

$y_2 = -\sin x \cos(\sin x) - \cos^2 x \sin(\sin x)$

In the I term of RHS multiply and divide

by $\cos x$.

$$y_2 = \frac{-\sin x}{\cos x} \cos x \cdot \cos(\sin x) - \cos^2 x \sin(\sin x)$$

$= -\tan x \, y_1 - \cos^2 x \, y$

$y_2 + \tan x \, y_1 + \cos^2 x \, y = 0.$

Hence the proof.

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(21) If $y = \sin(m \sin^{-1} x)$ then prove that

$(1-x^2) y - x y_1 + m^2 y = 0$

Given

$y = \sin(m \sin^{-1} x) \rightarrow (1)$

Differentiating on both sides with respect

to 'x' we get

$$y_1 = \frac{m \cos(m \sin^{-1} x)}{\sqrt{1-x^2}} \rightarrow (2)$$

$$\cancel{\sin}\ \frac{d}{dx}(m \sin^{-1}x) = \frac{m}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\, y_1 = m \cos(m \sin^{-1}x)$$

squaring on both sides we get

$$(1-x^2)\, y_1^2 = m^2 \cos^2(m \sin^{-1}x)$$

Differentiating again on both sides with respect to 'x' we get

$$(1-x^2)\, 2y_1 y_2 + y_1^2(-2x) = m^2 [2\cos(m \sin^{-1}x) \cdot \frac{d}{dx}(\cos(m \sin^{-1}x)]$$

Apply chain Rule on differentiating on Both

~~$(1-x^2)\ 2y_1 y_2 +$~~

$$= 2m^2 [\cos(m \sin^{-1}x) - \sin(m \sin^{-1}x) \frac{m}{\sqrt{1-x^2}}$$

$$= \frac{-2m^3 \cos(m \sin^{-1}x)}{\sqrt{1-x^2}} \sin(m \sin^{-1}x)$$

$$= -2m^2 \left[\frac{m \cos(m \sin^{-1}x)}{\sqrt{1-x^2}}\right] \sin(m \sin^{-1}x)$$

$$= -2m^2 (y_1)\, y$$

$(1-x^2)\, 2y_1 y_2 - 2xy_1^2 = -2m^2 y_1 y$

Dividing throught out by $2y_1$ we get

$(1-x^2)y_2 - xy_1 = -m^2 y$

$(1-x^2)y_2 - xy_1 + m^2 y = 0$

Hence the Result

(19) $y = \log(x+\sqrt{1+x^2})^2$ prove that $(1+x^2)y_2 + xy_1 = 2$

<u>Method-2</u>

$y' = 2\log(x+\sqrt{1+x^2})\dfrac{1}{x+\sqrt{1+x^2}}\left(1+\dfrac{\cancel{2}x}{\cancel{2}\sqrt{1+x^2}}\right)$

$= \dfrac{2\log(x+\sqrt{1+x^2})\,\cancel{(x+\sqrt{1+x^2})}}{\cancel{(x+\sqrt{1+x^2})}(\sqrt{1+x^2})}$

$y' = \dfrac{2\log(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}$

$\sqrt{1+x^2}\, y_1 = 2\log(x+\sqrt{1+x^2})$

squaring on both sides

$\because (2)^2 \underline{(\log(x+\sqrt{1+x^2})^2)}$ y

$(1+x^2)\, y_1^2 = 4y$

$(1+x^2)\, 2y_1 y_2 + y_1^2 (2x) = 4y'$

$(\div 2y_1)$

$(1+x^2)y_2 + xy_1 = 2.$

Hence proved -