2.1

Definitions :

i) $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ for $m>0,\ n>0$ is Known as Beta function and is denoted by $\beta(m,n)$

ii) $\int_0^\infty x^{n-1} e^{-x}\,dx$ for $n>0$ is Known as ~~Beta~~ Gamma function and is denoted by $\Gamma(n)$

2.2 Convergence of $\Gamma(n)$

$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx$; this integral exists if $n>0$.

$$\Gamma(n) = \int_0^1 x^{n-1} e^{-x}\,dx + \int_1^\infty x^{n-1} e^{-x}\,dx$$

The first integral is $\underset{\epsilon \to 0}{Lt} \int_\epsilon^1 x^{n-1} e^{-x}\,dx$ if this limit exists.

When x is Small, the integrand behaves like x^{n-1} and the limit axists if $n>0$.

The Second integral certainly exists for $e^{+x} > \frac{x^r}{r!}$ (r being any positive integer) $> \frac{x^{n+1}}{r!}$ so long as $r>n+1$

Hence $x^{n-1} e^{-x} < \frac{r!}{x^2}$

$\therefore \int_1^\infty e^{-x} x^{n-1}\,dx$ does not exceed a constant multiple of $\int_1^\infty \frac{dx}{x^2}$ which converges

$\therefore \Gamma(n)$ converges for $n>0$.

Cor. As we are going to see in 4, $\beta(m,n)$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$\beta(m,n)$ exists if $m>0$ and $n>0$.

2.3 Recurrence formula of Gamma function.

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\, dx \quad (n > -1)$$

Integrating it by parts taking $u = x^n$ and dv as $e^{-x}\,dx$, we get,

$$\Gamma(n+1) = \left[-e^{-x} x^n\right]_0^\infty - n\int_0^\infty -e^{-x} x^{n-1}\, dx$$

$$\underset{x\to 0}{Lt}\; e^{-x} x^n = 0 \text{ if } n > 0$$

$$\therefore \underset{x\to\infty}{Lt}\; e^{-x} x^n = \underset{x\to\infty}{Lt}\; \frac{x^n}{e^x} = 0$$

$$\therefore \Gamma(n+1) = n\int_0^\infty e^{-x} x^{n-1}\, dx$$

$$= n\,\Gamma(n) \text{ if } n > 0.$$

Note: This recurrence formula is true only when $n > 0$.

Cor: i: $\Gamma(n+1) = n!$ when n is a positive integer from the recurrence formula we have

$$\Gamma(n+1) = n\,\Gamma(n)$$

$$= n(n-1)\,\Gamma(n-1)$$

$$= n(n-1)(n-2)\ldots 1\,\Gamma(1)$$

$$\Gamma(1) = \int_0^\infty x^0 e^{-x}\, dx.$$

$$= \int_0^\infty e^{-x}\, dx$$

$$= \left[-e^{-x}\right]_0^\infty$$

$$\therefore \Gamma(n+1) = n!$$

Cor. (ii) $\Gamma(n+a) = (n+a-1)(n+a-2)\ldots a\,\Gamma(a)$

When n is a positive integer.

3. properties of Beta functions:

i) $\beta(m,n) = \beta(n,m)$.

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

putting $x = 1-y$, we have

$$\beta(m,n) = \int_1^0 (1-y)^{m-1} y^{n-1}(-dy)$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy$$

$$= \beta(n,m)$$

[This is merely a property of integrals, viz.]

$$\left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

ii) $\beta(m,n)$ can be expressed as a definite integral with $0, \infty$ as limits.

In $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, put $x = \frac{y}{1+y}$

When $x = 0$, $y = 0$; when $x = 1$, $y = \infty$

$$\text{Hence } \beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1}} \frac{1}{(1+y)^{n-1}} \frac{dy}{(1+y)^2}$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$$

iii) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1} x \cos^{2n-1} x \,dx$.

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

putting $x = \sin^2\theta$, we have

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

Which can be written as $2\, I_{2m-1,\, 2n-1}$.

$$\therefore I_{m,n} = \frac{1}{2}\beta\left[\frac{m+1}{2}, \frac{n+1}{2}\right]$$

4. Relation between Beta and Gamma functions

(1)
$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$\Gamma(m) = \int_0^\infty x^{m-1} e^{-x}\, dx$$

putting $x = t^2$ we have

$$\Gamma(m) = \int_0^\infty (t^2)^{m-1} e^{-t^2}\, 2t\, dt.$$

$$= 2\int_0^\infty t^{2m-1} e^{-t^2}\, dt.$$

So, we can take $\Gamma(m)$ as $2\int_0^\infty x^{2m-1} e^{-x^2}\, dx$.

$$\Gamma(n) = 2\int_0^\infty y^{2n-1} e^{-y^2}\, dy$$

$$\therefore \Gamma(m)\Gamma(n) = 4\int_0^\infty x^{2m-1} e^{-x^2}\, dx \int_0^\infty y^{2n-1} e^{-y^2}\, dy$$

$$= 4\int_0^\infty \int_0^\infty x^{2m-1} y^{2n-1} e^{-x^2-y^2}\, dx\, dy.$$

putting $x = r\cos\theta$, $y = r\sin\theta$, we have $dx\, dy = r\, dr\, d\theta$. x and y vary from 0 to ∞, x, y may be taken as the co-ordinates of any point in the first quadrant. By putting $x = r\cos\theta$, $y = r\sin\theta$ we are transforming cartesian into polar, the first quadrant can be covered by taking limits from 0 to ∞ for r and 0 to $\pi/2$ for θ.

$$\therefore \Gamma(m)\Gamma(n) = 4\int_0^\infty \int_0^{\pi/2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1} e^{-t^2}\, r\, dr\, d\theta.$$

$$= 4\int_0^\infty \int_0^{\pi/2} e^{-t^2} r^{2n+2m-1} \sin^{2m-1}\theta \cos^{2n-1}\theta\, dr\, d\theta.$$

$$= 4\int_0^\infty e^{-t^2} r^{2n+2m-1}\, dr \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta.$$

Now, $\int_0^\infty e^{-t^2} r^{2m+2n-1} dr = \frac{1}{2}\int_0^\infty t^{m+n-1} e^{-t} dt$ by putting $r^2 = t$

$$= \frac{1}{2}\Gamma(m+n)$$

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = F_{2m-1,\, 2n-1} = \frac{1}{2}\beta(m,n)$$

$$\therefore \Gamma(m)\Gamma(n) = 4 \cdot \frac{1}{2}\Gamma(m+n) \cdot \frac{1}{2}\beta(m,n)$$

$$= \Gamma(m+n)\beta(m,n)$$

$$\therefore \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Aliter. $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx.$

putting $x = xz$, where z is a positive constant with respect to x.

$$\Gamma(n) = \int_0^\infty e^{-xz} x^{n-1} z^n dx.$$

$$\Gamma(n).\, e^{-z} z^{m-1} = \int_0^\infty e^{-z(1+x)} z^{m+n-1} x^{n-1} dx.$$

$$\Gamma(n)\int_0^\infty e^{-z} z^{m-1} dz = \int_0^\infty \left[\int_0^\infty e^{-z(1+x)} z^{m+n-1} dz\right] x^{n-1} dx$$

putting $z(1+x) = y$, we get,

$$\int_0^\infty e^{-z(1+x)} z^{m+n-1} dz = \frac{1}{(1+x)^{m+n}} \int_0^\infty e^{-y} y^{m+n-1} dy.$$

$$= \frac{\Gamma(m,n)}{(1+x)^{m+n}}$$

Hence $\Gamma(n)\Gamma(m) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx \, \Gamma(m+n)$

$$= \Gamma(m+n)\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx.$$

$= \Gamma(m+n)\beta(m,n)$ from cor. (ii) of 3.

(2) cor1. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

put $m = n = \frac{1}{2}$

$$\therefore \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)} = \beta(\tfrac{1}{2}, \tfrac{1}{2})$$

$$\beta(\tfrac{1}{2}, \tfrac{1}{2}) = 2\int_0^{\pi/2} \sin^0\theta \, \cos^0\theta \, d\theta.$$

$$= 2\int_0^{\pi/2} d\theta = \pi$$

$$\Gamma(1) = 1$$

$$\therefore \left[\Gamma(\tfrac{1}{2})\right]^2 = \pi$$

Hence $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Cor(ii) In $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$. put $m = 1-n$;

then $\Gamma(n)\Gamma(1-n) = \beta(n, 1-n) = \int_0^\infty \dfrac{x^{n-1}}{1+x}\,dx$ cor(ii)

$$= \frac{\pi}{\sin n\pi}$$ (We shall assume this result)

if we put $n = \frac{1}{2}$ $\left\{\Gamma(\frac{1}{2})\right\}^2 = \dfrac{\pi}{\sin \pi/2} = \pi$; hence

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

Cor (iii) the result of 3 cor (iii)

3. $$\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta = \frac{1}{2}\beta(m,n)$$ is often expressed is the following form.

putting $2m = p$ and $2n = q$.

$$\int_0^{\pi/2} \sin^{p-1}\theta \, \cos^{q-1}\theta \, d\theta = \frac{1}{2}\beta\left[\frac{p}{2}, \frac{q}{2}\right]$$

$$= \frac{1}{2} \frac{\Gamma(p/2)\Gamma(q/2)}{\Gamma\left(\frac{p+q}{2}\right)} \dashrightarrow (1)$$

If we put $q=1$ in (1), we get

$$\int_0^{\pi/2} \sin^{p-1}\theta \, d\theta = \frac{1}{2} \frac{\Gamma(p/2)\Gamma(1/2)}{\Gamma\left(\frac{p+1}{2}\right)} \quad \text{---}\rightarrow (2)$$

by $\S$ 4 cor (i)

if we put $p=q$ in (1), we get

$$\int_0^{\pi/2} \sin^{p-1}\theta \cos^{p-1}\theta \, d\theta = \frac{1}{2} \frac{(\Gamma(p/2))^2}{\Gamma(p)}$$

i.e., $$\frac{1}{2^{p-1}} \int_0^{\pi/2} \sin^{p-1} 2\theta \, d\theta = \frac{\Gamma\{(p/2)\}^2}{2\Gamma(p)}$$

putting $2\theta = \varphi$, we get

$$\frac{1}{2^{p-1}} \int_0^{\pi} \sin^{p-1}\varphi \, d\varphi = \frac{\{\Gamma(p/2)\}^2}{\Gamma(p)}$$

i.e, $$\frac{2}{2^{p-1}} \int_0^{\pi/2} \sin^{p-1}\varphi \, d\varphi = \frac{(\Gamma(p/2))^2}{\Gamma(p)}$$

Using (2), we get

$$\frac{\sqrt{\pi}}{2^{p-1}\Gamma\left(\frac{p+1}{2}\right)} = \frac{\Gamma(p/2)}{\Gamma(p)}$$

Whence $$\Gamma(p/2)\,\Gamma\left(\frac{p+1}{2}\right) = \frac{\sqrt{\pi}}{2^{p-1}}\Gamma(p)$$

putting $p = 2n$, we have

$$\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2n)}{2^{2n-1}}$$

put $n = \frac{1}{4}$; $$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\sqrt{\pi}\,\Gamma(1/2)}{2^{-1/2}}$$

$$= \sqrt{2}\,\pi$$

5. We can evaluate certain definite integrals by using Gamma functions. the following examples will illustrate the method.

Examples.

Ex : 1 Evaluate $\int_0^1 x^m\left[\log \frac{1}{x}\right]^n dx$

put $\log\left(\frac{1}{x}\right) = t$, i.e, $x = e^{-t}$

$\therefore dx = -e^{-t}\, dt$

$$\int_0^1 x^m\left(\log \frac{1}{x}\right)^n dx = \int_0^\infty (e^{-t})^m t^n (-e^{-t}\, dt)$$

$$= \int_0^\infty e^{-(m+1)t}\, t^n\, dt.$$

put $(m+1)\, t = y$, $dt = \frac{1}{m+t}\, dy$.

Than the integral on this substitution becomes

$$\int_0^\infty \frac{e^{-y} y^n}{(m+n)^n} \frac{1}{m+n}\, dy = \frac{1}{(m+1)^{n+1}} \Gamma(n+1)$$

Ex : 2 $\int_0^\infty e^{-x^2} dx$.

put $x^2 = t$, i.e., $2x\,dx = dt$

i.e., $dx = \frac{1}{2} \cdot \frac{1}{\sqrt{t}}\, dt$.

$$\therefore \int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-t} \frac{1}{2\sqrt{t}}\, dt.$$

$$= \frac{1}{2}\int_0^\infty t^{-1/2} e^{-t}\, dt.$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{\pi}}{2}$$

Ex : 3 Evaluate

i) $\int_0^1 x^7 (1-x)^8\, dx$.

ii) $\int_0^{\pi/2} \sin^7\theta \cos^5\theta\, d\theta$.

iii) $\int_0^{\pi/2} \sin^{10}\theta\, d\theta$.

iv) $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$.

i) $\int_0^1 x^7(1-x)^8\,dx = \beta(8,9)$

$$= \frac{\Gamma(8)\Gamma(9)}{\Gamma(17)}$$

$$= \frac{7!\,8!}{16!}$$

(ii) $\int_0^{\pi/2} \sin^7\theta \cos^5\theta \, d\theta = \frac{1}{2}\beta\left[\frac{7+1}{2}, \frac{5+1}{2}\right]$

$$= \frac{1}{2}\beta(4,3)$$

$$= \frac{1}{2}\frac{\Gamma(4)(3)}{\Gamma(7)}$$

$$= \frac{1}{2}\cdot\frac{3!\,2!}{6!}$$

$$= \frac{1}{120}$$

(iii) $\int_0^{\pi/2} \sin^{10}\theta \, d\theta.$

$$= \frac{1}{2}\frac{\Gamma(11/2)\Gamma(1/2)}{\Gamma(11/2+1/2)} \quad \text{by 4, cor 3 (i)}$$

$$= \frac{\frac{1}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{\Gamma(6)}$$

$$= \frac{9\cdot7\cdot5\cdot3\,(\sqrt{\pi})^2}{5!\,2^6}$$

$$= \frac{63\pi}{512}$$

(iv) $\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta = \int_0^{\pi/2}\sin^{1/2}\theta\cos^{-1/2}\theta\,d\theta.$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left[\left(\frac{3}{4}+\frac{1}{4}\right)\right]} \quad \text{by §4, cor 3(i)}$$

$$= \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

$$= \frac{1}{2}\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

$$= \frac{\pi}{2\sin\frac{\pi}{4}} \quad \text{by § 4, cor (ii)}$$

$$= \frac{\pi}{\sqrt{2}}$$

Ex: 4 Express $\int_0^1 x^m(1-x^n)^p dx$ in terms of Gramma functions and evaluate and integral $\int_0^1 x^5(1-x^3)^{10}dx$.

put $x^n = y$, then $nx^{n-1}dx = dy$.

$$\therefore \int_0^1 x^m(1-x^n)^p dx = \int_0^1 y^{m/n}(1-y)^p \frac{dy}{n.y^{n-1/n}}$$

$$= \frac{1}{n}\int_0^1 y^{\frac{m-n+1}{n}}(1-y)^p dy.$$

$$= \frac{1}{n}\beta\left[\frac{m-n+1}{n}+1,\ p+1\right]$$

$$= \frac{1}{n}\beta\left[\frac{m+1}{n},\ p+1\right]$$

$$= \frac{1}{n}\frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma(p+1)}{\Gamma\left[\frac{m+1}{n}+p+1\right]}$$

$$\int_0^1 x^5(1-x^3)^{10}dx = \frac{1}{3}\frac{\Gamma\left(\frac{5+1}{3}\right)\Gamma(10+1)}{\Gamma\left(\frac{5+1}{3}+10+1\right)}$$

$$= \frac{1}{3}\frac{\Gamma(2)\Gamma(11)}{\Gamma(13)} = \frac{1}{396}.$$

Ex 5. prove

putting $\tan\theta$

$\int_0^\infty$

putting $\sqrt{b}$

to. $\frac{1}{2a^m}$

Ex 5. prove that $\int_0^{\pi/2} \frac{\cos^{2m-1}\theta \sin^{2n-1}\theta \, d\theta}{(a\cos^2\theta + b\sin^2\theta)^{m+n}} = \frac{\beta(m,n)}{2a^m b^n}$

Putting $\tan\theta = t$ the integral reduces to

$$\int_0^\infty \frac{t^{2n-1}\,dt}{(a+bt^2)^{m+n}}$$

putting $\sqrt{b}\,t = \sqrt{ay}$ the above integral reduces

to, $\frac{1}{2a^m b^n}\int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} = \frac{\beta(m,n)}{2a^m b^n}$