Find the n^{th} derivatives of co-efficient of $\log(x^2+a^2)$

sol:-

Given $\log(x^2+a^2) = \log(x+ia)+\log(x-ai)$

$D^n \log(x^2+a^2) = D^n \log(x-ai) + D^n \log(x+ai)$

use the formula.

$$\log(ax+b) = \frac{(-1)^{n-1}(n-1)!\, a^n}{(ax+b)^n}$$

$\therefore a=1 \Leftarrow \log(+x-ai)$

$b=ai$

$D^n \log(x^2+a^2) = D^n \log(x-ai) + D^n \log(x+ai)$

$$= \frac{(-1)^{n-1}(n-1)!\, 1^n}{(x-ai)^n} + \frac{(-1)^{n-1}(n-1)!\, 1^n}{(x+ai)^n}$$

simplify the above expression we apply Demoivre's theorem by taking.

$x = r\cos\theta$ and $a = r\sin\theta$

Now,

$x - ai = r\cos\theta - ir\sin\theta \Rightarrow r(\cos\theta - i\sin\theta)$

$x + ai = r(\cos\theta + i\sin\theta)$

$$D^n \log(x^2+a^2) = \frac{(-1)^{n-1}(n-1)!}{(x-ai)^n} + \frac{(-1)^{n-1}(n-1)!}{(x+ai)^n}$$

$$= (-1)^{n-1}(n-1)!\left[\frac{1}{(x-ai)^n} + \frac{1}{(x+ai)^n}\right]$$

$$= (-1)^{n-1}(n-1)!\left[\frac{1}{r^n(\cos\theta - i\sin\theta)^n} + \frac{1}{r^n(\cos\theta + i\sin\theta)^n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{r^n}\left[(\cos\theta - i\sin\theta)^{-n} + (\cos\theta + i\sin\theta)^{-n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{r^n}\left[\cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{r^n}(2\cos n\theta) \rightarrow (1)$$

$x = r\cos\theta$; $a = r\sin\theta$

$$\frac{a}{x} = \frac{r\sin\theta}{r\cos\theta} \Rightarrow \tan\theta$$

$\theta = \tan^{-1}(a/x)$

$$x^2+a^2 = r^2(\cos^2\theta+\sin^2\theta)$$

$$x^2+a^2 = r^2.$$

$$r = \sqrt{x^2+a^2}$$

sub in ①

$$D^n \log(x^2+a^2) = \frac{(-1)^{n-1}(n-1)!}{\left(\sqrt{x^2+a^2}\right)^n}\, 2\cos n\tan^{-1}(x/a)$$

Find the n^{th} derivatives of $y = x\log\left(\frac{x-1}{x+1}\right)$

$$y = x\log\left(\frac{x-1}{x+1}\right)$$

$$y = x\log(x-1) - x\log(x+1)$$

$$y_1 = x\frac{1}{x-1} + \log(x-1) - \log(x+1) - x\frac{1}{x+1}$$

$$= \frac{x}{x-1} + \log(x-1) - \log(x+1) - \frac{x}{x+1}$$

$$= \frac{x+1-1}{x-1} + \log(x-1) - \log(x+1) - \frac{x+1-1}{x+1}$$

$$= \frac{x-1}{x-1} + \frac{1}{x-1} + \log(x-1) - \log(x+1) - \frac{x+1}{x+1} + \frac{1}{x+1}$$

$$= \cancel{1} + \frac{1}{x-1} + \log(x-1) - \log(x+1) \cancel{-1} + \frac{1}{x+1}$$

$$= \frac{1}{x-1} + \frac{1}{x+1} + \log(x-1) - \log(x+1)$$

Differentiating both sides $(n-1)$ times with respect to 'x' we get

$$y_n = D^{n-1}\left(\frac{1}{x-1}\right) + D^{n-1}\left(\frac{1}{x+1}\right) + D^{n-1}\log(x-1) - D^{n-1}\log(x+1)$$

$$= \frac{(-1)^{n-1}(n-1)!}{(x-1)^n} + \frac{(-1)^{n-1}(n-1)!}{(x+1)^n} + \frac{(-1)^{n-2}(n-2)!}{(x-1)^{n-1}} - \frac{(-1)^{n-2}(n-2)!}{(x+2)^{n-1}}$$

14/10/20

10M

※ LEIBNITZ THEOREM

Leibnitz Theorem ~~produce~~ provides the n^{th} derivatives of the produce of 2 functions. It states at follows.

If $y = uv$ where u and v are function of x and their n^{th} derivatives are seperately known, then

$$\frac{1}{n} = (uv)_n = u_n v + nC_1 u_{n-1} v_1 + nC_2 u_{n-2} v_2 + \cdots$$
$$nC_r u_{n-r} v_r + u v_n$$

(ie) $$D^n(uv) = u_n v + nC_1 u_{n-1} v_1 + nC_2 u_{n-2} v_2 +$$
$$nC_r u_{n-r} v_r + \cdots + u v_n$$

2) Find the n^{th} derivatives of $x^2 \sin 3x$.

Sol :-

$u = \sin 3x$ $\qquad$ $v = x^2$

$u_n = 3^n \sin\left(3x + \frac{n\pi}{2}\right)$ $\qquad$ $v_1 = 2x$

$u_{n-1} = 3^{n-1} \sin\left(3x + (n-1)\frac{\pi}{2}\right)$ $\qquad$ $v_2 = 2$

$u_{n-2} = 3^{n-2} \sin\left(3x + (n-2)\frac{\pi}{2}\right)$ $\qquad$ $v_3 = 0$.

$$D^n(x^2 \sin 3x) = x^2\, 3^n \sin\left(3x + \frac{n\pi}{2}\right) + nC_1\, 3^{n-1} \sin\left(3x + \frac{(n-1)\pi}{2}\right) \cdot 2x + nC_2\, 3^{n-2} \sin\left(3x + (n-2)\frac{\pi}{2}\right) \cdot 2$$

$$= x^2 3^n \sin\left(3x + \frac{n\pi}{2}\right) + \frac{n!}{(n-1)!\,1!} 3^{n-1} \sin\left(3x + \frac{(n-1)\pi}{2}\right) 2x + \frac{n!}{(n-2)!\,2!} 3^{n-2} \sin\left(3x + \frac{(n-2)\pi}{2}\right) 2.$$

$$= x^2 3^n \sin\left(3x + \frac{n\pi}{2}\right) + \frac{n(n-1)}{(n-1)!} 3^{n-1} \sin\left(3x + \frac{(n-1)\pi}{2}\right) 2x + \frac{n(n-1)(n-2)}{(n-2)\,2!} 3^{n-2} \sin\left(3x + \frac{(n-2)\pi}{2}\right) 2$$

1×2

$$= x^2 3^n \sin\left(3x + \frac{n\pi}{2}\right) + 2x\, n\, 3^{n-1} \sin\left(3x + \frac{(n-1)\pi}{2}\right) + n(n-1)\, 3^{n-2} \sin\left(3x + \frac{(n-2)\pi}{2}\right)$$

3) If $y = e^{a \sin^{-1} x}$ S.T (i) $(1-x^2)y_2 - xy_1 = a^2 y$

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$.

Sol:-

Given $y = e^{a \sin^{-1} x}$

$$y_1 = e^{a \sin^{-1} x} (a) \cdot \frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\, y_1 = e^{a \sin^{-1} x} \cdot a$$

$$\sqrt{1-x^2}\, y_1 = ay$$

Squaring on both sides

$$(1-x^2) y_1^2 = a^2 y^2$$

Differentiating again on both sides

$$(1-x^2)\, 2y_1 y_2 + y_1^2 (-2x) = a^2 2yy'$$

$\div\ 2y'$

$$(1-x^2)\, y_2 - xy_1 = a^2 y$$

Hence the proof

Differentiates n term with using Leibnitz theorem.

$$\left[(1-x^2)y_{n+2}+nC_1 y_{n+1}(-2x)+nC_2 y_n(-2)\right]-$$

$$(xy_{n+1}+nC_1 y_n(1)] = a^2y_n$$

$$(1-x^2)y_{n+2}-2xny_{n+1}-n(n-1)y_n - xy_{n+1}-ny_n = a^2y_n$$

$$(1-x^2)y_{n+2}-2xn\,y_{n+1}-n^2y_n+\cancel{ny_n}-xy_{n+n}-\cancel{ny_n} = a^2y_n$$

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n-a^2y_n=0$$

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+a^2)y_n=0.$$

4) $y=(\sin^{-1}x)^2$ then prove that

(i) $(1-x^2)y_2-xy=2$

(ii) $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0$

Given $y=(\sin^{-1}x)^2$

$$y_1 = 2\sin^{-1}x\,\frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,y_1 = 2\sin^{-1}x.$$

Squaring on both sides.

$$(1-x^2)y_1^2 = 4(\sin^{-1}x)^2$$

$$(1-x^2)y_1^2 = 4y.$$

Differentiating again on both sides

$(1-x^2)\, 2y_1 y_2 + y_1^2(-2x) = 4y_1$

$(1-x^2)\, 2y_1 y_2 - 2x y_1^2 = 4y_1$

$\div\, 2y_1$

$(1-x^2)y_2 - xy_1 = 2$ (or) $(1-x^2)y_2 - xy_1 - 2 = 0$

Hence proved

Differentiate n term with using Leibnitz theorem.

$[(1-x^2)\, y_{n+2} + nc_1\, y_{n+1}(-2x) + nc_2\, y_n(-2)] -$

$(xy_{n+1} + nc_1\, y_n] = 0$

$(1-x^2)y_{n+2} - 2nx\, y_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n = 0$

$(1-x^2)\, y_{n+2} - 2nx y_{n+1} - n^2 y_n + ny_n - xy_{n+1} - ny_n = 0$

$(1-x^2)\, y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$

Increasing and Decreasing function.

A function $f(x)$ is said to be an increasing function if the value of $f(x)$ increasing as x is increasear.

Similarly, $f(x)$ is said to be decreasing function if the value of $f(x)$ decreasens as x is decreasens.

Ex;

$\sin x$ is an increasing function for $0 < x < \pi/2$

$\cos x$ is an decreasing. function for $0 < x < \pi/2$

Conditions:-

If $f(x)$ is an increasing function when $F'(x) > 0$

If $f(x)$ is a decreasing. function when $f'(x) < 0$.

Monotonic function:-

A function $f(x)$ is said to be monotonic function if it is either increasing (or) decreasing throw out

10) S.T if x is positive $x - \cancel{x^3}/6 < \sin x < x$ for $x > 0$

Sol:-

Given ; $x - \frac{x^3}{6} < \sin x < x$.

$$F(x) = \sin x - \left(x - \frac{x^3}{6}\right)$$

$$F'(x) = \cos x - \left(1 - \frac{\cancel{3}x^2}{\cancel{6}_2}\right).$$

$$= \cos x - 1 + \frac{x^2}{2}$$ will be > 0

$F'(x) > 0$.

$f(x)$ is an increasing function.

$$\sin x - \left(x - \frac{x^3}{6}\right) > 0$$

$$\sin x > \left(x - \frac{x^3}{6}\right) \rightarrow 1$$

$$g(x) = \sin x - x$$

$$g'(x) = \cos x - 1$$ will be < 0

$g'(x) < 0$

$g(x)$ is an decreasing function.

$g(x) < 0$

$\sin x - x < 0$

$\sin x < x. \rightarrow 2$

From 1 & 2

$$x - \frac{x^3}{6} < \sin x < x$$

f(x) is an increasing function

P.T $\dfrac{1-2x-x^2}{1+x-2x^2}$ always decrease an, x?

in increasing function.

Sol:-

$$F(x)=\frac{1-2x-x^2}{1+x-2x^2}$$

$$F'(x)=\frac{(1+x-2x^2)(-2-2x)-(1-2x-x^2)(1-4x)}{(1+x-2x^2)^2}$$

$$=\frac{(-2-2x+4x^2-2x-2x^2+4x^3)-[1-2x-x^2-4x+8x^2+4x^3]}{(1+x-2x^2)^2}$$

$$=\frac{(-2-4x+2x^2+4x^3)-(1-6x+7x^2+4x^3)}{(1+x-2x^2)^2}$$

$$=\frac{-2-4x+2x^2+4x^3-1+6x-7x^2-4x^3}{(1+x-2x^2)^2}$$

$$F'(x) = \frac{-5x^2+2x-3}{(1+x-2x^2)^2}$$

$$F'(x) = \frac{-(5x^2-2x+3)}{(1+x-2x^2)^2} < 0$$

$F(x)$ is an decreasing function

P.T $\log(1+x)$ lies between $x - \frac{x^2}{2}$ and $x - \frac{x^2}{2(1+x)}$, $x > 0$

Sol:-

$$x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}$$

$$F(x) = \log(1+x) - \left(x - \frac{x^2}{2}\right)$$

$$F'(x) = \frac{1}{1+x} - \left(1 - \frac{2x}{2}\right)$$

$$= \frac{1}{1+x} - (1-x)$$

$$= \frac{1-(1-x^2)}{1+x} \Rightarrow \frac{x^2}{1+x} > 0$$

$F(x) > 0$ $F(x)$ is an increasing

$f(x) > 0 \Rightarrow \log(1+x) - \left(x - \frac{x^2}{2}\right) > 0$

$\log(1+x) > x - \frac{x^2}{2} \rightarrow ①$

$$g(x) \Rightarrow \log(1+x) - \left(x - \frac{x^2}{2(1+x)}\right)$$

$$g'(x) = \frac{1}{1+x} - \left[1 - \frac{\cancel{2}x}{\cancel{2}(1)}\right]$$

$$= \frac{1}{1+x} - (1-x)$$

$$= \frac{1-1+x^2}{1+x} = \frac{x^2}{1+x}$$

$g'(x) < 0$ $g'(x)$ is an decreasing

$$g'(x) < 0 \Rightarrow \log(1+x) - \left(x - \frac{x^2}{2(1+x)}\right) < 0$$

$$\log(1+x) < x - \frac{x^2}{2(1+x)}$$