UNIT - I

Chapter - 3:

The general second order linear differential equation is of the form,

$y'' + p(x)y' + Q(x)y = R(x) \longrightarrow (1)$

Here, $p(x), Q(x), R(x)$ are func. of x alone or constant.

The eqns. (1) cannot be solved explicity in terms of known elementary func are even in terms of indicated integration.

Existence and uniqueness theorem:

Let $p(x), Q(x), R(x)$ be a continuous function on the $[a,b]$. If x_0 is any point in $[a,b]$ and if y_0 and y_0' are any numbers second order linear differentiation equation.

$y(x)$ on the indicated interval such that,

$y(x_0) = y_0$ and $y'(x_0) = y_0'$

Theorem: [proof need not be necessary]

Statement:

If y_g is the general soln of $y'' + p(x)y' + Q(x)y = 0$ and y_p is the particular soln. of the equation,

$y'' + p(x)y' + Q(x)y = R(x) \longrightarrow (1)$

Then, The complete soln. of (1) is given by

$$\boxed{y_g = y_p + y_c}$$

Theorem:

If $y_1(x)$ and $y_2(x)$ are any two soln. of,

$$y'' + p(x)y' + Q(x)y = 0 \longrightarrow (1)$$

Then,

$c_1 y_1(x) + c_2 y_2(x)$ is also a soln. for any constants c_1 and c_2.

Trivial solution:

$y(x) = 0$ always a solution.

$$y'' + py' + Qy = 0$$

This is called trivial soln. The soln. $c_1 y_1(x) + c_2 y_2(x)$ is called a linear combination of the soln. of $y_1(x)$ and $y_2(x)$.

Linear Dependent:

If two ~~line~~ functions $f(x)$ and $g(x)$ are defined on an interval $[a, b]$ and have the property that one is a constant multiple of other, then they are said to be linear dependent on $[a, b]$.

Linear Independent:

If neither is the constant multiple of other then they are called linearly independent linearly dependent $f(x) = 0$. Then, $f(x)$ and $g(x)$ are linearly dependent for every func. $g(x) = 0$ order of $f(x)$.

If two function $x_1(t)$ & $x_2(t)$ defined in I is said to be linearly dependent on I.

If there exists two constants c_1 & c_2. Atleast one of them is non-zero. There exists

$$c_1(x_1)t + c_2(x_2)t = 0.$$

Wrokskain Determinants:

2m The wronkskain of any two differentiation function $y_1(x)$ and $y_2(x)$ is given by.

$y_1 y_2' - y_2 y_1'$ is defined as

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' \neq 0.$$

Theorem:

Consider $y_1(x)$ and $y_2(x)$ be linearly independent soln. of homogenous eqn.

$$y'' + p(x)y' + Q(x)y = 0 \longrightarrow (1)$$

on the interval $[a, b]$; Then, to prove

$c_1 y_1(x) + c_2 y_2(x)$ is the general solution of the second order linear eqn. on $[a, b]$, in the such that, every soln. second order linear equation.

on this interval can be obtained from (1).

By the suitable choice by arbitrary constant c_1 and c_2.

NOTE:

c^* also if wronkskain determinant is not equal to zero. Then the function are linearly independent.

If 2 function, $x_1(t)$, $x_2(t)$ are linearly independent.

and if $c_1 x_1(t) + c_2 x_2(t) = 0 \in I$. Then,

$c_1 = 0$, $c_2 = 0$.

proof:

$y_1(x)$ and $y_2(x)$ be the two linearly independent.

Soln. of eqn. (1)

$$y'' + y'p(x) + Q(x)y = 0.$$

To prove:

$c_1 y_1(x) + c_2 y_2(x)$ is general soln. of (1).

Let,

$y(x)$ be any soln. of (1) on $[a, b]$.

constants c_1 and c_2 can be found,

$y(x)$.

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \quad \forall \; x \in [a, b]$$

By known theorem,

[Existence of Unique theorem]

A soln. of (1) over all the value of its derivatives at a single point. It is a sufficient to show that,

$c_1 y_1(x) + c_2 y_2(x)$ and $y(x)$ are both soln. of (1) on $[a, b]$ for some x_0 in $[a, b]$.

We can find c_1 and c_2, so that,

$$c_1 y_1(x_0) + c_2 y_2(x_0) = y(x_0)$$

$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = y'(x_0)$$

To solve c_1 and c_2 is sufficient that the determination.

$$W(y_1(x_0), y_2(x_0)) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \neq 0.$$

Have, a value of different from zero from the above it is noted. The function of x is defined by,

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

Which is known as wronskain of y_1 and y_2.

The Necessary and Sufficient Conditions are,

Lemma: 1

The Necessary part,

If $y_1(x)$ and $y_2(x)$ are two soln. of eqns.(1)

$[y''+p(x)y'+Q(x)y=0]$ on $[a,b]$. Then their

Wronkskain. $y''+p(x)y'+Q(x)y=0 \longrightarrow (1)$

$W=W(y_1,y_2)$ is either identically, zero or

never zero on $[a,b]$. $W=0$ Lin Dep

proof: $W\neq 0$, Lin Independent

We know that,

By known Wronskain method,

$$w = y_1 y_2' - y_2 y_1' \longrightarrow (2)$$

UV method $= uv'+vu'$

Then, $w' = dw/dx = y_1 y_2'' + \cancel{y_2' y_1'} - y_2 y_1'' - \cancel{y_1' y_2'}$

$$\frac{dw}{dx} = y_1 y_2'' - y_2 y_1'' \longrightarrow (3)$$

since, put, $y=y_1$; $y=y_2$ in (1),

y_1 and y_2 are both sides. of (1) we have

$$y_1''+p(x)y_1'+Q(x)y_1=0 \longrightarrow (4)$$

$$y_2''+p(x)y_2'+Q(x)y_2=0 \longrightarrow (5)$$

Now,

Multiplying (4) by y_2 and (5) by y_1.

we get,

eqn. $(4)\times y_2 \Rightarrow y_1'' y_2 + p(x) y_1' y_2 + Q(x) y_1 y_2 = 0 \longrightarrow (6)$

$(5)\times y_1 \Rightarrow y_2'' y_1 + p(x) y_1 y_2' + Q(x) y_1 y_2 = 0 \longrightarrow (7)$

Eqn. (7) - (6),

$$y_2'' y_1 - y_1'' y_2 + P y_2' y_1 - P y_1' y_2 + Q y_1 y_2 - Q y_1 y_2 = 0$$

$dw/dx + P(w) = 0$

$$(y_2'' y_1 - y_1'' y_2 + P(y_1 y_2' - y_1' y_2)) = 0$$

From (2) and (3), we have,

$$dw/dx + pw = 0$$

This is the first order linear dependent eqn. where general soln. is,

$$w = ce^{-\int}$$ $\frac{dw}{dx} = -Pw$

$$\boxed{W = ce^{-\int p dx}}$$ $w = ce^{-\int p dx}$

The exponential factor, the exponential is never zero.

$$\boxed{y_2(x) = k y_1(x)} \quad \forall \; x \in [a, b]$$ The constant $C = 0$ and never

The conclude the lemma, if $C \neq 0$ and

with this lemma the proof of the main theorem is complete.

To eliminate C, we have,

$$\begin{vmatrix} y_2 & -y_1 \\ y_2' & -y_1' \end{vmatrix} = -y_2 y_1' + y_1 y_2' = 0$$

$$y_1 y_2' - y_2 y_1' = 0$$

$$W(y_1 - y_2) = 0$$ is identically zero

$W = 0$

Lemma: 2

The sufficient part, linearly dependent

Conversely we assume that, constant multiple.

$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$ is identically zero.

<u>To prove:</u>

$y_1(x)$ and $y_2(x)$ are linearly dependent. If y_1 is identically zero.

Assume that,

y_1 does not vanish identically on $[a, b]$ from which it follows be.

continuity. that y_1 does not vanish.
At some sub internal $[c, d]$ and $[a, b]$

$$y_1 y_2' - y_2 y_1' = 0 \text{ on } [a, b]$$

divided by y_1^2,

$$y_1 y_2' - y_2 y_1' = 0$$

$$\Rightarrow \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = 0$$

$$\Rightarrow \frac{y_1 y_2'}{y_1^2} - \frac{y_2 y_1'}{y_1^2} = 0$$

$$\Rightarrow (y_2/y_1)' = 0$$

$$\frac{y_2'}{y_1} - \frac{y_2 y_1'}{y_1^2}$$

$$\frac{y_1 y_2' - y_2 y_1'}{y_1^2}$$

zero

On integrating we get,

$y_2 / y_1 = K$ (Constant form)

$y_2 = K y_1$ where K is constant.

$\Rightarrow y_2(x) = K y_1(x) \ \forall \ x \in [c, d]$

Finally, $y_2(x)$ and $K y_1(x)$ has equal values in $[c, d]$. Then, Have equal derivatives there also known by 1st theorem,

$$y_2(x) = K y_1(x) \ \forall \ x \in [a, b].$$

This conclude the lemma with this lemma the proof of the main theorem is complete.

Working Rule:

* To find the two soln. of an eqn.

$$y'' + Py' + Qy = 0 \text{ (integrated)}$$

An integral is to show that their ratio is not constant. This is done by inspection.

* Employ the format test is complete.

Wronkskain $W(y_1, y_2)$ and prove that its does not vanish not equal to zero.

PROBLEMS :

1. prove that $y = c_1 \sin x + c_2 \cos x$ in the general soln. of $y_1'' + y_1 = 0$ on any interval. find particular solution for which $y(0) = 2$, $y'(0) = 3$.

$y(x_0) = y_0$

$x_0 = 0,$

Soln: Given, $y = c_1 \sin x + c_2 \cos x$

is the general soln.

$$y = c_1 y_1(x) + c_2 y_2(x)$$

we have,

$$y_1(x) = \sin x$$
$$y_1'(x) = \cos x$$
$$y_1''(x) = -\sin x$$

Suppose,

y_1 is a soln. of given equation, we have,

$$y_1'' + y_1 = -\sin x + \sin x = 0$$
$$\therefore y_1'' + y_1 = 0$$

If y_2 is a soln.

$$y_2'' + y_2 = -\cos x + \cos x = 0$$
$$\therefore y_2'' + y_2 = 0$$

consider,

$$y_1 / y_2 = \frac{\sin x}{\cos x} = \tan x.$$

$y_1 = c y_2$ [y_1 and y_2 are not linearly independent]

using,

Wronkskain Method we have,

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$
$$= -\sin^2 x - \cos^2 x$$
$$= -(\sin^2 x + \cos^2 x)$$

$\Rightarrow -1 \Rightarrow -1 \neq 0$.

$\therefore$ The soln. y_1 and y_2 are linearly independent.

Let us take $y'' + y = 0$. $\because y'' + P(x)y' + Q(x)y = 0$

Here, $P = 0$ and $Q = 1$ are constant functions.

By a known theorem,

$y = C_1 \sin x + C_2 \cos x$ is a general solution of $y'' + y = 0$.

To find particular soln.:-

Let us take $y(x) = C_1 \sin x + C_2 \cos x$.

$y_1(0) = \sin 0 = 0$

putting, $x = 0 \Rightarrow y(0) = C_2$. $y_2(0) = \cos 0 = 1$

But, it is given that,

$y(0) = 2 \Rightarrow C_2 = 2$.

$y'(x) = C_1 \cos x - C_2 \sin x$

$y'(0) = C_1$

But, It is Given that,

$y'(0) = 3 \Rightarrow C_1 = 3$.

The particular soln. is

$\therefore y = 3 \sin x + 2 \cos x$

2) Show that e^x and e^{-x} are linearly independent Soln. of $y'' - y = 0$ on an interval.

soln:

Let us take,

$y = C_1 e^x + C_2 e^{-x}$ is of the form,

$y = C_1 y_1(x) + C_2 y_2(x)$

$y_1 = e^x \qquad y_2 = e^{-x}$

$y_1' = e^x \qquad y_2' = -e^{-x}$

$y_1'' = e^x \qquad y_2'' = e^{-x}$

Suppose,
y_1 is the soln. $y'' - y = 0 \longrightarrow (1)$

Then,
$y_1'' - y_1 = e^x - e^x = 0 \longrightarrow (2)$

Suppose,
y_2 is the soln. $y'' - y = 0$

Then,
$y_2'' - y_2 = e^{-x} - e^{-x} = 0 \longrightarrow (3)$

From (2) and (3), we have,
$y_1 = e^x,\ y_2 = e^{-x}$ are solutions of the eqn. (1).

To prove they are L.I.:
Let us consider,
Wronkskain determinant,

$$W = (y_1 - y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

$$= -e^x e^{-x} - e^x e^{-x} = -2e^x e^{-x}$$

$$= -2e^0 = -2$$

$W(y_1; y_2) = -2 \quad \therefore W(y_1 - y_2) = -2 \neq 0.$

y_1 and y_2 are not linearly dependent. Let y_1 and y_2 are linearly independent soln. of $y'' - y = 0$. Hence proved.

3) Show that $y = c_1x + c_2x^2$ is the general soln. of $x^2y'' - 2xy' + 2y = 0$. on any interval and find partiaular solution for which $y(1) = 3$ and $y'(1) = 5$.

Soln: Let us take, $y = c_1x + c_2x^2$ and is of the form, $y = c_1y_1(x) + c_2y_2(x)$

$y_1(x) = x$ | $y_2(x) = x^2$

$y_1' = 1$ | $y_2' = 2x$

$y_1'' = 0$ | $y_2'' = 2$