from (2) and (3) it is clear that

$y_1 = e^x$ and $y_2 = e^{2x}$ is the soln of

$y'' - 3y' + 2y = 0$

To find particular soln

Let us take, $y(x) = c_1 e^x + c_2 e^{2x}$

$y'(x) = c_1 e^x + 2c_2 e^{x}$

putting, $x = 0$

$\Rightarrow c_1 e^0 + c_2 e^0 + c$

$\Rightarrow c_1 + c_2$

Given that,

$y(0) = -1$

$c_1 + c_2 = -1 \rightarrow (4)$

putting, $x = 0$ is $y'(x) = c_1 e^x + 2c_2 e^{2x}$

$y'(0) = c_1 + 2c_2$

Given that

$y'(0) = 1$

$c_1 + 2c_2 = 1 \rightarrow (5)$

Equ (4) - (5) $\Rightarrow$ $c_1 + c_2 = -1$

$-c_1 - 2c_2 = -1$

$-c_2 = -2$

$c_2 = 2.$

Substitute (2) in (4)

$c_1 + 2 = -1 \Rightarrow -3.$

The particular soln is

$y(x) = -3e^x + 2e^{2x}$

5) S.T e^{2x} and xe^{2x} and linearly independent soln of $y'' - 4y' + 4y = 0$ in an interval

Soln: Let us take

$y = c_1 e^{2x} + c_2 x e^{2x}$ is of the form

$y = c_1 y_1(x) + c_2 y_2(x)$

$y'' - 4y' + 4y = 0 \rightarrow (1)$

$y_1 = e^{2x}$ $\qquad$ $y_2 = xe^{2x}$

$y_1' = 2e^{2x}$ $\qquad$ $y_2' = 2xe^{2x} + e^{2x}$

$y_1'' = 4e^{2x}$ $\qquad$ $y_2'' = 4xe^{2x} + 2e^{2x} + 2e^{2x}$

$= 4e^{2x} + 4xe^{2x}$

Suppose, y_1 is the soln of (1)

$y_1'' - 4y_1' + 4y_1$

$\Rightarrow 4e^{2x} - 8e^{2x} + 4e^{2x} = 8e^{2x} - 8e^{2x}$

$y_1'' - 4y_1' + 4y_1 = 0 \rightarrow (2)$

Suppose, y_2 is the soln of (1)

$y_2'' - 4y_2' + 4y_2 = 4xe^{2x} + 4e^{2x} - 8xe^{2x} - 4e^{2x} + 4xe^{2x}$

$y_2'' - 4y_2' + 4y_2 = 0 \rightarrow (3)$

from (2) and (3) It is clear that

$y_1 = e^{2x}$ and $y_2 = xe^{2x}$ is the soln of

$y'' - 4y' + 4y = 0$

To prove they are L.I:

Let us consider, wronskain determinant

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix}$$

$$= e^{2x}(2xe^{2x} + e^{2x}) - xe^{2x}(2e^{2x})$$

$$= \cancel{2xe^{4x}} + e^{4x} - \cancel{2xe^{4x}}$$

$W(y_1, y_2) = e^{4x} \neq 0.$

$\therefore y_1$ and y_2 are L.I soln of

$y'' - 4y' + 4y = 0.$

Section - 16

The use of known soln to find another if we know two linearly independent soln $y_1(x)$ and $y_2(x)$. Then we can find the general soln of the

$y'' + P(x)y' + Q(x)y = 0 \rightarrow (1)$

... the standard procedure for determining y_2, when y_1 is known,

Assume that,

$y_1(x) \neq 0$ is known soln of (1) They, To develop the

...dure we assume that

$y_1(x) \neq 0$ is known soln of (1) we [know] that

$cy_1(x)$ is also a soln for any constant c. Here, the basic idea is a replace constant c by an unknown func $v(x)$ and then to determine v in such a way that $y_2 = vy_1$ will be a soln of $y(1)$.

$$y_2'' + p(x)y_2' + Q(x)y_2 = 0 \rightarrow (2)$$

Then, determine the unknown func $v(x)$

$$y_2 = vy_1$$

Then,

$$y_2' = vy_1' + y_1v' \quad (\text{uv method})$$

$$y_2'' = vy_1'' + y_1'v' + y_1v'' + v'y_1'$$

$$y_2'' = vy_1'' + 2y_1'v' + y_1v''$$

Then, $y_2'' + p(x)y_2' + Q(x)y_2 = 0$

becomes,

$$vy_1'' + y_1v'' + 2y_1'v' + pvy_1' + py_1v' + Qvy_1 = 0$$

$$v(y_1'' + py_1' + Qy_1) + v'(2y_1' + py_1) + v''y_1 = 0$$

Since,

y_1 is a soln of (1) then coefficient of v, becomes zero $[\because v = 0]$

$$(3) \Rightarrow v'(2y_1' + py_1) + v''y_1 = 0$$

$$v''y_1 = -v'(2y_1' + py_1)$$

$$\frac{v''}{v'} = -\frac{1}{y_1}(2y_1' + py_1)$$

$$\frac{v''}{v'} = -\frac{2y_1'}{y_1} - p. \quad \left\{ \int \frac{1}{x} = \log x \right\}$$

$\int$ing both sides we get

$$\log v' = -2\log y_1 - \int p\,dx$$

$$= -\log y_1^2 - \int p\,dx$$

$$\log v' + \log y_1^2 = -\int p\,dx$$

$$\Rightarrow \log v' y_1^2 = -\int p\,dx$$

Taking exponential on both sides we get

$$v'y_1^2 = e^{-\int p\,dx}$$

and $$v' = \frac{1}{y_1^2} e^{-\int p\,dx}$$

$\int$ ing both sides $v = \int \frac{1}{y_1^2} e^{-\int p dx} dx$

we have to S.T y_1 and y_2

$v = \int \frac{1}{y_1^2} e^{-\int p dx} dx$ are linearily independent

problems:

If y_1 is a non-zero soln of second order linear equ and $y_2 = v y_1$ where given by $v = \int \frac{1}{y_1^2} e^{-\int p dx} dx$ is the second soln found interval allow by completing the wronkskain that y_1 and y_2 are linearily independent

Soln Given that,

y is a non zero soln of

$y'' + py' + qy = 0$

$y_2 = v y_1$ where,

$v = \int \frac{1}{y^2} e^{-\int p dx}$

we have to show that y_1 and y_2 are linearily independent soln

To complete wronkskain,

we have

$w(y_1 y_2) = y_1 y_2' - y_2 y_1'$

$= y_1 (v y_1' + v' y_1) - v y_1 y'$

$= v y_1 y_1' + v' y_1^2 - v y_1 y_1'$

$= v' y_1^2$

$= \frac{1}{y_1^2} e^{-\int p dx} y_1^2$

$w(y_1, y_2) = e^{-\int p dx}$

The exponential will never zero

ie, $w(y_1, y_2) \neq 0$

$\therefore y_1(x)$ and $y_2(x)$ are L.I soln

2) PT if $y_1(x) = x$ is a soln of $x^2 y'' + xy' - y = 0$ which is simple enough to be discovered by inspection find general soln.

Sol: Given that

$y_1(x) = x$ is soln of given D.E

$x^2 y'' + xy' - y = 0 \rightarrow (1)$

$\div x^2$

$\Rightarrow y'' + \frac{y'}{x} - \frac{y}{x^2} = 0 \rightarrow (2)$

$$v = \int \frac{1}{y_2} e^{-\int p\,dx}\, dx$$

$$= \int \frac{1}{x^2} e^{-\int \frac{-2x}{1-x^2} dx}\, dx$$

$$= \int \frac{1}{x^2} e^{-\log(1-x^2)}\, dx$$

$$= \int \frac{1}{x^2} e^{\log(1-x^2)^{-1}}\, dx$$

$$v = \int \frac{1}{x^2(1-x^2)}\, dx \longrightarrow (4)$$

consider

$$\frac{1}{x^2(1-x^2)} = \frac{A}{x^2} + \frac{B}{1-x^2}$$

$$1 = A(1-x^2) + Bx^2$$

Put $x = 1$

$$1 = A(1-1) + B(1) \qquad \boxed{B=1}$$

Put $x = 0$

$$\Rightarrow 1 = A(1) + B(0)$$

$$\boxed{A=1}$$

$$\frac{1}{x^2(1-x^2)} = \frac{1}{x^2} + \frac{1}{1-x^2} \longrightarrow (5)$$

Substitute the eqn (5) in (4)

$$v = \int \left(\frac{1}{x^2} + \frac{1}{1-x^2} \right) dx$$

$$= \int \frac{dx}{x^2} + \int \frac{dx}{1-x^2}$$

$$= \int x^{-2} dx + \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$$

$$= \frac{x^{-1}}{-1} + \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$$

$$v = \frac{-1}{x} + \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$$

$$y_2 = v y_1 = -\frac{1}{\cancel{x}} \cancel{x} + \frac{x}{2} \log\left(\frac{1+x}{1-x}\right)$$

$$y_2 = -1 + \frac{x}{2} \log\left(\frac{1+x}{1-x}\right)$$

The general soln is given by

$$y = c_1 y_1 + c_2 y_2$$

$$y = c_1 x + c_2 \left[\frac{x}{2} \log\left(\frac{1+x}{1-x}\right)^{-1} \right]$$

4) use $y_1 = x$ is an soln of following equ find general soln.

$$y'' - \frac{x}{(x-1)} y' + \frac{1}{x-1} y = 0$$

$y'' - \frac{x}{(x-1)} y' + \frac{1}{x-1} y = 0$

$P = \frac{-x}{x-1}$ $Q = \frac{1}{x-1}$

second order linear independent soln is given by $y_2 = v y_1$

$v = \int \frac{1}{y_1^2} e^{-\int P dx} dx$

$\int \frac{1}{x^2} e^{-\int \frac{-x}{x-1} dx} dx$

$\int \frac{1}{x^2} e^{\int \frac{x+1-1}{x-1} dx} dx$

$= \int \frac{1}{x^2} e^{\int (\frac{x+1}{x-1} + \frac{1}{x-1}) dx} . dx$

$= \int \frac{1}{x^2} e^{\int (dx) + \int \frac{dx}{x-1}} dx$

$= \int \frac{1}{x^2} e^{x + \log(x-1)} dx$

$= \int \frac{1}{x^2} e^{x} e^{\log(x-1)}$

$= \int \frac{1}{x^2} e^{x(x-1)} dx$

$= \int (\frac{e^x . x}{x^2} \quad \frac{-e^x}{x^2}) dx$

$\int \frac{e^x}{x} dx - \int \frac{e^x}{x^2} dx$

$= \int e^x x^{-1} dx - \int e^x x^{-2} dx$

$V = x^{-1} e^x$

$= e^x / x$

$y_1 = v y_1 = \frac{1}{x} e^x . x$

$y_2 = e^x$.

The general soln is given by

$y = c_1 y_1 + c_2 y_2$

$\therefore y = c_1 x + c_2 e^x$.

ON THE METHOD OF VARIATION OF PARAMETERS:

Determine a particular soln of the non-homogeneous equ.

$y'' + P(x) y' + Q(x) y = R(x) \longrightarrow (1)$

has several limitations

consider The corresponding equ,

$y'' + P(x) y' + Q(x) y = 0 \rightarrow (2)$

Independent soln of the homogeneous eq

Let

$y(x) = c_1 y_1(x) + c_2 y_2(x)$ be the general soln of (2)

where,

c_1 and c_2 are constants, here we by unknown func, $v_1(x)$ and $v_2(x)$ and attempts of determine v_1 and v_2 is such a manner that

$$y = v_1 y_1 + v_2 y_2 \rightarrow (3)$$

will be the soln of (1)

[the parameter v_1 and v_2 are variables and hence the variation of parameters]

Differentiable (3) with respect to x,

we get, $y' = v_1 y_1' + y_1 v_1' + v_2 y_2' + y_2 v_2' \rightarrow (4)$

we can choose v_1 and v_2 such that,

$$v_1' y_1 + v_2' y_2 = 0 \rightarrow (5)$$

[we required only $v_1' + v_2'$ on if y'' given $v_1' + v_2''$ which is negligeble].

$(4) \Rightarrow y' = v_1 y_1' + v_2 y_2' \rightarrow (6)$

Again, differentiate with respect to x,

$$y'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2'' \rightarrow (7)$$

Substitute,

(3), (6), (7) in (1) we get

$v_1' y_1' + v_1 y_1'' + v_2' y_1' + v_2 y_2'' + P(x)[v_1 y_1' + v_2 y_2'] + Q(x)(v_1 y_1 + v_2 y_2) = R$

$v_1(y_1'' + P(x) y_1' + Q(x) y_1) + v_2(y_2'' + P(x) y_2' + Q(x) y_2) + v_1' y_1' + v_2' y_2' = R(x) \rightarrow$ (i)

$\because y_1$ and y_2 are linearily independent soln of (2) so we have,

$$y_1'' + P(x) y_1' + Q(x) y_1 = 0 = y_2'' + P(x) y_2' + Q(x) y_2$$

and,

we vanishes get,

$$v_1' y' + v_2' y_2' = R(x) \rightarrow (8)$$

Solving for v_1' and v_2' from (5) and (8)

we get $\quad v_1' y_1 + v_2' y_2 = 0 \rightarrow (A)$

$$v_1' y_1' + v_2' y_2' = R(x) \rightarrow (B)$$

$(A)\, xy_1' \Rightarrow v_1' y_1 y_1' + v_2' y_2 y_1' = 0$

$(B)\, xy_1 \Rightarrow v_1' y_1' y_1 + v_2' y_2' y_1 = R(x) y_1$

$(-) \quad (-) \quad (-)$

$$v_2'(y_2 y_1' - y_2' y_1) = R(x) y_1$$

$$v_2' = \frac{y_1(R(x))}{y_2 y_1' - y_1 y_2'}$$

Similarly

$v_1' = \dfrac{-R(x)\, y_2}{w(y_1, y_2)}$ and it is clear $w(y_1, y_2) \neq 0$.

$\because$ y_1 and y_2 are linear independent solns

$\int$ ing v_1' and v_2' we get,

$$V_1 = \int -\frac{y_2 R(x)}{w(y_1, y_2)}\, dx$$

$$V_2 = \int \frac{y_1 R(x)}{w(y_1, y_2)}\, dx$$

Thus the particular soln of (1) is given by (3) $\Rightarrow y = v_1 y_1 + v_2 y_2$

$$\therefore y = y_1 \int \frac{-y_2 R(x)}{w(y_1, y_2)}\, dx + y_2 \int \frac{y_1 R(x)}{w(y_1, y_2)}\, dx.$$

Note :

This method has some disadvantages integrals v_1 and v_2 may be difficult of impossible to workout.

It is necessary to know the general soln of equ (2) before the process can be started.

Example :

Find the variation of parameters particular soln of

$y'' + y = \operatorname{cosec} x$

Sol: let the homogenous equ be $y'' + y = 0$

Then its general soln of given by $(D^2+1)y = 0$

Auxillary equ is $m^2 + 1 = 0$

$\Rightarrow m^2 = -1$

$m = \pm i$

$C.F = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$= e^{0}(A\cos x + B\sin x)$

$C.F \quad y = c_1 \sin x + c_2 \cos x.$

General soln

$y = c_1 \sin x + c_2 \cos x$

$y_1 = \sin x \qquad y_2 = \cos x$

$y_1' = \cos x \qquad y_2' = -\sin x$

$y_1'' = -\sin x \qquad y_2'' = -\cos x$

By wronskain determinants

$w(y_1, y_2) = y_1 y_2' - y_2 y_1'$

$= \sin x(-\sin x) - \cos x \cos x$

$= -(\sin^2 x + \cos^2 x)$

$w(y_1, y_2) = -1 \neq 0$

we have,

$$v_2 = \int \frac{y_1 R(x)}{w(y_1, y_2)} dx$$

$$= \int \frac{\sin x \operatorname{cosec} x}{-1} dx$$

$$= \int -\frac{\sin x}{\sin x} dx = \int -dx$$

$$\boxed{v_2 = -x}$$

$$v_1 = \int \frac{-y_2 R(x)}{w(y_1, y_2)} = \int -\frac{\cos x \operatorname{cosec} x}{-1} dx$$

$$= \int \frac{\cos x}{\sin x} dx$$

$$\boxed{v_1 = \log \sin x}$$

we have the particular soln,

$$\boxed{y = y_1 v_1 + y_2 v_2}$$

$y = y_1(\log \sin x) + y_2(-x)$

$y = \sin x(\log \sin x) - x \cos x$

section - 27

2M series soln of first order equation power series an infinite series is of the form

$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ is called a power series in x

Definition:

The series $\sum a_n x^n$ is said to converge at a point x. If the limit $n \to \infty$. $\lim_{n\to\infty} \sum_{n=0}^{\infty} a_n x^n$ exist and the sum of the series is the value of this limit with respect to the corresponds of the point of convergence and radius of convergence.

for example:

consider the series $\sum_{n=0}^{\infty} n!\, x^n = 1 + x + x^2 + \dots$

this series diverges for all values of $x \neq 0$

consider the series $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots + \frac{1 \cdot x^n}{1-x}$ converges if $|x| < 1$

diverges $|x| > 1$

This kind of series corresponding to the real number R is called Radius of convergence such that the series converges if $|x| < R$ and the series diverges if $|x| > R$.

1) Find the Power series soln of the differentiable eqn $y' = y$ using power series.

Soln:

The power series soln of the form.

$y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$.

This can be diff eqn term by term on its interval of converges.

$y' = a_1 + a_2 2x + 3 a_3 x^2 + \dots + n a_n x^{n-1} + \dots + (n+1) a_{n+1} x^{n+1}$

$y_1 = y$ must have the same coefficient comparing the same co-efficient.

$a_0 = a_1,\ 2a_2 = a_1,\ 3a_3 = a_2 \dots$

$(n+1) a_{n+1} = a_n$

$a_2 = \frac{a_1}{2} = \frac{a_0}{2!} \qquad a_3 = \frac{a_2}{3}$

$= \frac{a_0}{3 \times 2!} = \frac{a_0}{3!}$

$a_{n+1} = \frac{a_0}{(n+1)!} = \frac{a_0}{n!(n+1)}$.

$y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$

becomes $y = a_0 + a_0 x + \frac{a_0}{2!} x^2 + \frac{a_0}{3!} x^3 + \dots + \frac{a_0}{(n+1)!} x^{n+1}$