$5a_5 = 3/2\cdot 4 \Rightarrow a_5 = 3/40$

$7a_7 = \frac{15}{6\cdot 8} \Rightarrow a_7 = \frac{15}{6\cdot 7\cdot 8} = \frac{1\cdot 3\cdot 5}{7\cdot 6\cdot 8} = \frac{1\cdot 3\cdot 5}{7(2\cdot 4\cdot 6)}$

$a_2 = 0;\ a_4 = 0;\ a_6 = 0$

by condition,

$y(0) = 0,\quad a_0 = 0$

$$y = x + \frac{x^3}{3(2)} + \frac{x^5}{5}\left(\frac{1\cdot 3}{2\cdot 4}\right) + \frac{x^7}{7}\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right) + \cdots$$

$$\sin^{-1} x = x + \sum_{n=1}^{\infty} \frac{x^{2n+1}}{2n+1}\left(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\right).$$

put $x = 1/2$

$$\sin^{-1}(1/2) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\ \frac{(1/2)^n}{2^{n+1}}$$

$$\pi/6 = 1/2 + 1/2 \cdot 1/3\cdot 2 + \cdots$$

Hence the proof.

4) Express the $\tan x = x + x^3/3 + 2/15\cdot x^5 + \cdots$ by solving equ $y' = 1+y^2$ with $y(0) = 0$.

Soln: let $y = \tan x \rightarrow (1)$

Differentiate (1) with respect to x

$y' = \sec^2 x = 1 + \tan^2 x = 1 + y^2$

$y' = 1+y^2$ where (1) is the power series of the foll. D.E

$y' = 1+y^2 \quad y(0) = 0 \rightarrow (2)$

we assume that (2) has a power series soln of the form

$y = \sum a_n x^n$

$y = a_0 + a_1 x + \cdots + a_n x^n \rightarrow (3)$

It follows that,

$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + 5a_5 x^{n-1} + (n+1) a_{n+1} x^n + \cdots$

$y' = 1+y^2$

$= 1 + [a_0 + a_1 x + \cdots + a_n x^n + \cdots]^2$

$= 1 + [a_0^2 + a_1^2 x^2 + a_2^2 x^4 + \cdots + a_n^2 x^{2n} + 2a_0 a_1 x_2 a_0 +$

$a_2 x^2 + \cdots + 2a_0 a_n x^n + \cdots$

$+ 2a_1 a_n x^n + 2a_2 a_3 x^4 + 2a_2 a_4 x^6 + \cdots$

$+ 2a_2 a_{n-2} + \cdots]$

Equating the corresponding coefficient with $y(0)=0$

$a_0=0,\ a_1=1\ 2a_2=2a_0a_1=0 \Rightarrow a_2=0$

$3a_3=a_1^2+2a_0-1a_2=a_1^2+0 \Rightarrow a_3=1/3$

$4a_4=0 \Rightarrow a_4=0$

$5a_5=a^2+2a_2a^4+2a_1a_3=0+0+2/3$

$a_5=2/15$

$7a_7=a_3^2+2a_0a_6+2a_1a_5+2a_2a_4$

$=1/9+0+2/15+0=11/45$

$a_7=11/315$

$y=x+1/3x^3+2/15x^5+\frac{11}{315}x^7+\ldots$

Section 2.5

Second order linear equs:

A pt x_0 is an ordinary point if both $p(x)$ & $q(x)$ are analytic at x_0. If o point is not ordinary is a singular point. of the power series: If $x_0 \subset (a,b)$ expanded fu analytic at $x=x_0$

$f(x)=\sum_{n=0}^{\infty} c_n(x-x_0)^n$

Ordinary points:

Consider the general homogenous 2nd order linear equs.

$y''+p(x)y'+q(x)y=0 \rightarrow (1)$

Here $p(x)$ & $q(x)$ are function of x a point. x_0 depend on the behaviour of it coefficent for $p(x)$ & $q(x)$ near this point.

$p(x)$ & $q(x)$ are analytic at x_0 which needs that each has a power series expansion valid in some neighborhood of this points.

In this case x_0 is called an ordinary point in (1)

It gives that every soln of the equ is also analytic at the point in other words of the coefficient of (1) at a certain point implies that it solns. are also analytic there any point that is not an ordinary point of (1) is called a singular point.

1) solve. D.E is $y''+y=0$ by power series method :

Sol: Given diff equ is $y''+y=0 \rightarrow (1)$

The coefficient func are

$$y''+p(x)y'+q(x)y=0$$

$$P(x)=0, \quad Q(x)=1$$

$\therefore$ These func are analytic at all points. so we see a soln of the form

$$y=\sum a_n x^n \longrightarrow (2)$$

$$y=a_0+a_1x+a_2x^2+\dots+a_nx^n+\dots$$

Diff w.r. to x

$$y'=a_1+2a_2x+3a_3x^2+\dots+na_nx^{n-1}+(n+1)a_{n+1}$$

$$y'=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n \longrightarrow (3)$$

$$y''=2a_2+6a_3x+12a_4x^2+\dots+n(n+1)a_{n+1}x^{n+1}+\dots$$

$$\dots+(n+1)(n+2)a_{n+2}x^n+\dots$$

$$y''=\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n \longrightarrow (4)$$

$$y''+y=0$$

$$2a_2+6a_3x+\dots(n+1)(n+2)a_{n+2}x^n+a_0+a_1x+\dots a_nx^n=0$$

Sum of the coefficient is equal to zero

$$(n+1)(n+2)a_{n+2}+a_n=0$$

$$a_{n+2}=\frac{-a_n}{(n+1)(n+2)}$$

this recursion formula given as express as according as n is even (or) odd.

If $n=0 \Rightarrow a_2=-a_0/2$

$n=1 \Rightarrow a_3=-a_1/6$

$n=2 \Rightarrow a_4=-a_2/12=-\frac{(-a_0/2)}{12}=a_0/24$

$n=3 \Rightarrow a_5=-a_3/4\cdot5=a_1/4\cdot5\cdot6=a_1/4\cdot5\cdot6$

with this coefficient of [2]

$$y=a_0+a_1x-a_0/2\,x^2-a_1/6\,x^3+a_0/24\,x^4+a_1/4\cdot5\cdot6\,x^5$$

collecting a_0, a_1

$y=[a_0-a_0$

$=a_0[1-x^2/$

$y=a_0$

te :

(5) satisfies (1

$a_0=1$ and $a_1=0$ Th

$y=a_0$

$y_0=a_0$

$y_1=co$

$y_1'=$

$y_1''=$

$y_1''+y_1=$

||ly. if $a=0$

$y=a_0y$

Wronskian Meth

To fin

dependent.

$w(y_1,y$

$y_1=$

$w(y$

y_1 and y

y

y_1 and

$\Rightarrow$

Let $y_1(x)=\cos$

there (5) is a gen

obtained by sp

collecting a_0, a_1 separately we have

$$y = [a_0 - a_0/2\, x^2 + a_0/24\, x^4 + \dots] +$$

$$[a_1 x - a_1/6\, x^3 + a_1/4\cdot5\cdot6\, x^5 + \dots]$$

$$= a_0 [1 - x^2/2 + x^4/24 + \dots] + a[x - x^3/3! + x^5/5! + \dots]$$

$$y = a_0 \cos x + a_1 \sin x \rightarrow (5)$$

$y = \cos x$

Note:

(5) satisfies (1) for any 2 constants a_0 and a_1 in particular if $a_0 = 1$ and $a_1 = 0$ Then,

$$y = a_0 y_1 + a_1 y_2$$

$$y_0 = a_0 \cos x + a_1 \sin x$$

$y_1 = \cos x$. satisfies (1)

$$y_1' = -\sin x$$

$$y_1'' = -\cos x$$

$$y_1'' + y_1 = -\cos x + \cos x = 0.$$

$||^{ly}$. if $a = 0$ and $a_1 = 1$

$$y = a_0 y_1 + a_1 y_2 = a_0 \cos x + a_1 \sin x$$

$y = \sin x$, satisfies (1)

Wronskian Method:

To find y_1, y_2 are whether linear independent or linear dependent.

$$w(y_1, y_2) = y_1 y_2' - y_0 y_1'$$

$$y_1 = \cos x \qquad y_2 = \sin x$$

$$w(y_1, y_2) = \cos x(\cos x) - \sin x(-\sin x)$$

$$= \cos^2 x + \sin^2 x$$

$$w(y_1, y_2) \neq 0.$$

y_1 and y_2 are linear equ

$$y_1/y_2 = \cot x$$

y_1 and y_2 are linearly independent

$\Rightarrow y_1 \neq c y_2$ [$\because$ (5) is the general soln of (1)]

let $y_1(x) = \cos x$ and $y_2(x) = \sin x$ are linear independent there (5) is a general soln. of (1) and that any series is obtained by specifying the value of $y(0) = a_0$ and $y'(0) = a_1$

Legendre equation n^{th} degree equation

1) Solve the second order Legendre eqn by power series method

Two forms of Legendre formula

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0 \longrightarrow (1)$$

The coeffi func are

$$P(x) = \frac{-2x}{1-x^2}, \quad q(x) = \frac{p(p+1)}{1-x^2} \longrightarrow (2)$$

are analytic origin $(1-x^2)y'' - 2xy' + p(p+1)y = 0$

origin is an ordinary point

Let us assume that the power series such that as

$$y = \Sigma a_n x^n.$$

$$y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \longrightarrow (3)$$

Also,

$$y' = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y'' = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$$

$$-x^2 y'' = -x^2 [2a_2 + 6a_3 x + \dots + (n+1)(n+2) a_{n+2} + \dots$$

$$= -[2a_2 x^2 + 6a_3 x^3 + \dots + (n-1) n a_n x^n + \dots]$$

$$-x^2 y'' = -\sum_{n=0}^{\infty} n(n-1) a_n x^n \longrightarrow (a)$$

$$-2xy' = -[2x a_1 + 4x^2 a_2 + 6x^3 a_3 + \dots + 2n a_n x^n + \dots]$$

$$= -\Sigma 2n a_n x^n$$

$$= -2\Sigma n a_n x^n \longrightarrow (b)$$

The sum of the series required to be a zero. So the coefficient of x^n must be zero for every n.

we have,

$$[(n+1)(n+2) a_{n+2} - n(n-1)a_n - 2n a_n + p(p+1) a_n] = 0$$

$$(n+1)(n+2) a_{n+2} = [n(n-1) a_n + 2n a_n - p(p+1) a_n]$$

$$= [n(n-1) + 2n - p(p+1)] a_n$$

$$= [n^2 - n + 2n - p^3 - p] a_n$$

$(+1)$ and (-1) in by $pn = [n^2 + n - p^2 - p]\, a_n$

$$\Rightarrow [n^2 + n - p^2 - p - pn + np] a_n$$

$$\Rightarrow -[-n^2 - n + p^2 + p - pn + pn] a_n$$

$$\Rightarrow -[p^2 + p(1+n) - pn - n(n+1)] a_n$$

$$\Rightarrow -[p^2 - pn + p(1+n) - n(n+1)] a_n$$

$$\Rightarrow -[p(p-n) + (n+1)(p-n)] a_n$$

$$a_{n+2} = \frac{-(p-n)(n+1+p)}{(n+1)(n+2)} \cdot a_n$$

The Recursion formula enable us to express a_n in terms of a_0 (or) a_1, accordingly as interme of a_0 (or) a_1, accordingly as n is even (or) odd

$n=0$, $a_2 = \frac{-p(p+1)}{2!} a_0$

$n=1$, $a_3 = \frac{-(p-1)(p+2)}{3!} a_1$

$n=2$, $a_4 = \frac{-(p-2)(p+3)}{3 \cdot 4} a_2$

$= \frac{-(p-2)(p+3)}{3 \cdot 4} \times \frac{(-p(p+1))}{2} a_0$

$n=3$, $a_5 = \frac{(p-3)(p-1)(p+2)(p+4)}{5!} a_1$

$n=4$, $a_6 = \frac{-(p-4)(p-2)p(p+1)(p+3)(p+5)}{6!} a_0$

$n=5$, $a_7 = \frac{(p-5)(p-3)(p-1)(p+2)(p+4)(p+6)}{7!} a_1$

By intersecting these coefficient in (3) we get

$$y = \left[a_0 + a_1 x - \frac{p(p+1)}{2!} a_0 x^2 - \frac{(p-1)(p+2)}{3!} a_1 x^3 + \cdots - \frac{(p-1)(p+1)(p+3)}{4!} a_0 x^4 + \cdots\right]$$

$$y = a_0\left[1 - \frac{p(p+1)}{2!} x^2 + \frac{(p-2)(p+3)(p+1)}{4!} x^4 + \cdots\right] + a_1\left[x - \frac{(p-1)(p+2)}{3!} x^3 + \frac{(p-3)(p-1)(p+2)(p+4)}{5!} x^5 + \cdots\right] \to (5)$$

(5) is the formula soln for (1)

$\Rightarrow y = a_0 y_1 + a_1 y_2$

where y_1 and y_2 are,

$$y_1 = \left[1 - \frac{p(p+1)}{2!} x^2 + \frac{(p-2)(p+3)(p+1)}{4!} x^4 + \cdots\right]$$

$$y_2 = \left[x - \frac{(p-1)(p+2)}{3!} x^3 + \cdots\right]$$

The func. defined by 5 6 is called legendge funct

Case i) when p is not an integers each series, the 1st and 2nd series has radius of convergence

$$a_{n+2} = \frac{-(p-n)(n+1+p)}{(n+1)(n+2)} a_n.$$

Ratio Test: Put $n = 2n$

$$a_{n+2} = \frac{-(p-2n)(2n+1+p)}{(2n+1)(2n+2)} a_{2n}$$

$$\Rightarrow \frac{a_{2n+2}}{a_{2n}}$$

$$\left| \frac{a_{2n+2}\, x^{2n+2}}{a_{2n}\, x^{2n}} \right| \leq \left| \frac{-(p-2n)(2n+1+p)}{(2n+1)(2n+2)} \right|$$

$|x^2|$ as $n \to \infty$

It is similar for the 2nd series (5) is the general soln of (1) on the interval $|x|<1$ because the series are linearly independent.

case ii) :

If p is non-negative integer one of the series terminates and therefore is a polynomial. [The 1st series is a polynomial if p is even and 2nd series is a polynomial if p is odd]

The other series is an infinite series

① Find the general soln of $(1+x^2)y'' + 2xy' - 2y = 0$ in terms of power series in x can you express that soln by means of elementary func.

Soln: Let,

$$(1+x^2)y'' + 2xy' - 2y = 0 \longrightarrow (1)$$

$$P(x) = \frac{2x}{(1+x^2)}, \quad Q(x) = \frac{-2}{1+x^2}$$

are analytical origin. origin is an ordinary point

Let us assume that following power series

$$y = \sum a_n x^n$$

$$y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots \rightarrow (3)$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots + n a_n x^{n-1} + \dots$$

$$y' = \sum (n+1) a_{n+1} x^n \longrightarrow (4)$$

$$y'' = 2a_2 + 6a_3 x + \dots + (n+1)(n+2) a_{n+2} x^n + \dots$$

$$= \sum (n+1)(n+2) a_{n+1} x^n \longrightarrow (5)$$

$$x^2 y'' = 2a_2 x^2 + 6a_3 x^3 + \dots + (n-1)^n a_n x^n + \dots$$

$$= \sum n(n-1) a_n x^n \longrightarrow (6)$$

$$2xy' = 2x a_1 + 4a_2 x^2 + 6a_3 x^3 + \dots + 2n a_n x^n$$

$$= 2 \sum a_n x^n \longrightarrow (7)$$

$-2y = -[2a_0 + 2a_1 x + 2a_2 x^2 + \dots + 2a_n x^n + \dots]$

$= 2 \sum a_n x^n$

The sum of the series required to be zero, so that coefficient of x^n must be zero for every n.

$\therefore (n+1)(n+2) a_{n+2} + (n-1) a^n + 2 a a_n - 2a_n = 0$

$(n+1)(n+2) a_{n+2} = a_n [-n^2 + n - 2n + 2]$

$$a_{n+2} = \frac{a_n [-n^2 - n + 2]}{(n+1)(n+2)}$$

$$a_{n+2} = \frac{-a_n (n^2 + n - 2)}{(n+1)(n+2)}$$

using recursion formula, we express all a_n term of a_0 and a_1.

If $n=0 \Rightarrow a_2 = \frac{+a_0}{2} (+2) = a_0$

$n=1 \Rightarrow a_3 = \frac{-a_2}{2 \cdot 3} (0) = 0$

$n=2 \Rightarrow a_4 = \frac{-a_2}{3 \cdot 4} (4) = -a_0/3$

$n=3 \Rightarrow a_5 = -\frac{a_3}{4 \cdot 5} (0) = 0$

$n=4 \Rightarrow a_6 = \frac{a_4}{5 \cdot 6} (8) = \frac{a_0 \times 18}{6 \times 5 \times 3} = a_0/5$

$n=5 \Rightarrow a_7 = \frac{-a_5}{6 \cdot 7} (2 \cdot 8) = \frac{-a_1 (0)}{7} = 0$

By intersecting these coefficient in (3), we have.

$y = a_0 + a_1 x + a_2 x^2 + \dots a_n x^n + \dots$

$= a_0 [1 + x^2 + (-x^4/3) + x^6/4 + \dots] + a_1 x$

$= a_0 [1 + x(x - x^3/3 + x^5/5 + \dots] + a_1 x$

$y = a_0 [1 + x \tan^{-1} x] + a_1 x$

2) consider the equation $y'' + xy' + y = 0$, find the general soln $y = \sum a_n x^n$ is the form of $y = a_0 [y_1(x) + a_1 y_2(x)]$ where $y_1(x)$ & $y_2(x)$ are power series.

Proof: let $y'' + xy' + y = 0$

Here, $P(x) = x$, $Q(x) = 1$. are analytic at origin

- Let us assume that following. power series

$y_n = \sum a_n x^n$

$y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots + (n+1) a_{n+1} x^n + \dots$

$= \Sigma (n+1)(a_{n+1}) x^n$

$y'' = 2a_2 + 6a_3 x + \cdots + (n+2)(n+1)a_{n+2} x^n$

$= \Sigma (n+2)(n+1) a_{n+2} x^n$

$xy' = a_1 x + 2a_2 x^2 + \cdots + n a_n x^n$

$= \Sigma n a_n x^n$

The sum of the series assumed to be zero. The coefficient of x^n must be zero for every n.

$(n+2)(n+1) a_{n+2} + n a_n + a_n = 0$

$(n+1)(n+2) a_{n+2} = -a_n (n+1)$

$a_{n+2} = \dfrac{-a_n}{a_{n+2}}$

$n=2, \; a_4 = -a_2/4 = a_0/8$

$n=3, \; a_5 = -a_3/5 = a_1/15$

$n=4, \; a_6 = -a_4/6 = -a_0/48$

$n=5, \; a_7 = -a_5/7 = -a_0/103$

By inserting the coefficient in ② we get,

$y = a_0 + a_1 x - a_0/2 \, x^2 - a_1/3 \, x^3 + a_0/3 \, x^4 + a_1/15 \, x^5$

$y = a_0 [1 - x^2/2 + x^4/3 - x^6/48 + \cdots] +$

$a_1 [x - x^3/3 + x^5/15 - x^7/3\cdot5\cdot7]$

Theorem:

Let x_0 be an ordinary point of the diff eqn $y'' + p(x) y' + Q(x) y = 0$. Let a_0 be arbitrary constant of a unique func $y(x)$.

(i.e.) Analytic x_0 is a soln of (1) in a certain nbd of the point & satisfies the initial condition $y(x_0) = a_0, y'(x_0) = a_1$

Proof Given,

$y'' + p(x) y' + Q(x) y = 0 \longrightarrow$ ①

For our convenience

Let us take, $x_0 = 0$ the power series in x rather than $(x - x_0)$

By Hypothesis

The theorem $P(x)$ & $Q(x)$ are analytic at origin and therefore they have power series expansion

$$p(x) = \sum_{n=0}^{\infty} p_n x^n = p_0 + p_1 x + p_2 x^2 + \cdots \rightarrow (2)$$

$$Q(x) = \sum_{n=0}^{\infty} q_n x^n = q_0 + q_1 x + q_2 x^2 + \cdots \rightarrow (3)$$

$$|x| < R, \quad R > 0$$

There,

p and q converge on the interval $|x| < R$ for some $R > 0$ with the given initial condition we find a soln. for (1) in the form of power series,

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \rightarrow (4)$$

which satisfies the initial condition $y(0) = a_0$ and $y'(0) = 0$.

with radius of convergence R and since $|x| < R$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots \rightarrow (5)$$

$$y'' = 2a_2 + 6a_3 x + \cdots$$

$$= \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n \longrightarrow (6)$$

It follows that from the rule of multiply the power series.

$$p(x) y' = \left(\sum_{n=0}^{\infty} p_n x^n\right)\left(\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n\right)$$

$$= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{\infty} p_{n-k} (k+1) (a_{k+1})\right] x^n \rightarrow (7)$$

$$Q(x) y = \left(\sum_{n=0}^{\infty} q_n x^n\right)\left(\sum_{n=0}^{\infty} a_n x^n\right)$$

$$= \sum_{n=0}^{\infty} (q_{n-k} a_k) x^n \longrightarrow (8)$$

On subs (6), (7), (8) in (1) and additing the term by term we get

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} x^n + \sum_{k=0}^{n} p_{n-k} (k+1) a_{k+1} x \right] + \left[\left(\sum_{k=0}^{n} q_{n-k} a_k\right) x^n\right] = 0$$

$$(n+1)(n+2) a_{n+2} + \sum_{k=0}^{\infty} p_{n-k} (k+1) a_{k+1} + \sum_{k=0}^{\infty} q_{n-k} a_k = 0$$

Sohe the we have to following recursion formula

$$(n+1)(n+2) a_{n+2} = -\sum_{k=0}^{n} \left[(k+1) p_{n-k} a_{k+1} + q_{n-k} a_k \right]$$

For $n = 0, 1, 2, \ldots$ the above formula

becoms $n=0$ $k=0$ sub in equ. (9)

$$2a_2 = -\left[p_0 (0+1) a_0 + 1 + q_0 a_0 \right]$$

$2a_2 = -[p_0 a_1 + a_0 a_0]$

$n=1$ $k=0,1$ sub in ①

$6a_3 = -[p_1 a_1 + q_1 a_0 + 2p_0 a_2 + a_0 a_1]$

$n=2$ $k=0,1,2,$ sub in ①

$12a_4 = -[p_2 a_1 + q_2 a_0 + 2p_1 a_2 + q_1 a_1 + 3p_0 a_3 + a_0 a_2]$

The above formula determine $a_2, a_3 \ldots$ interms of a_0 and a_1, the resulting series (4) which satisfies (1) & the given critical condition is uniquely determined the requirement.

3) verify that the equation $y'' + y' - xy = 0$ has a 3 term recursion formula & find its series solution $y_1(x), y_2(x)$ show that

a) $y_1(0) = 1$, $y_1'(0) = 0$

b) $y_2(0) = 0$, $y_2'(0) = 1$

Sol:

let, $y'' + y' - xy = 0 \longrightarrow$ ①

$P(x) = 1$, $Q(x) = x \longrightarrow$ ②

are analytic at origin

Let, $y = \sum a_n x^n$

$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

$= \sum a_n x^n \longrightarrow$ ③

$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (n-1)(a_{n+1}) x^n$

$= \sum (n+1) a_{n+1} x^n \longrightarrow$ ④

$y'' = 2a_2 + 6a_3 x + 12 a_4 x^2 + \cdots + (n+2)(n+1) a_{n+2} x^n$

$= \sum (n+1)(n+2) a_{n+2} x^n$

the sum of the series required to be zero, so that, coefficient of x^n must be zero for ~~enough~~ every n.

$(n+1)(n+2) a_{n+2} + (n+1) a_{n+1} - a_{n-1} = 0$

$(n+1)(n+2) a_{n+2} = a_{n-1} - (n+1)(a_{n+1})$

$$a_{n+2} = \frac{a_{n-1} - (n+1)(a_{n+1})}{(n+1)(n+2)}$$

If. $n=0 \Rightarrow a_2 = -a_1/2$ $\qquad = \dfrac{a_n - (n+1) a_n}{(n+1)(n+2)}$

$n=1 \Rightarrow a_3 = -\frac{a_0}{2\cdot 3} + \frac{a_1}{2\cdot 3}$

$n=2 \Rightarrow a_4 = +\frac{a_1}{3\cdot 4} - \frac{a_3}{4}$

$y''+y'-xy=0$

$(n+1)(n+2)a_{n+2}-$

$(n+1)a_{n+1}-na_n$

$= a_1/3\cdot4 - a_0/2\cdot3\cdot4$

$= a_1/2\cdot3\cdot4$

$$a_4 = \frac{a_1 - a_0}{2\cdot3\cdot4}$$

$xy' = n a_n x^n$

$x^2y'' = n(n-1)a_n x^n$

$$n=3 \Rightarrow a_5 = \frac{a_2}{4\cdot5} - \frac{a_4}{5}$$

$$= \frac{a_1}{2\cdot4\cdot5} - \frac{a_1}{3\cdot4\cdot5} + \frac{a_0}{2\cdot3\cdot4\cdot5}$$

$$= \frac{-4a_1}{2\cdot3\cdot4\cdot5} + \frac{a_0}{2\cdot3\cdot4\cdot5}$$

Insecting the coefficient on (1) we get

$$y = a_0 + a_1x + \frac{a_1}{2}x^2 + \left[\frac{a_0}{2\cdot3} + \frac{a_1}{2\cdot3}\right]x^3 + \left[\frac{a_1}{2\cdot3\cdot4} - \frac{a_0}{2\cdot3\cdot4}\right]$$

$$- \left[\frac{a_0}{2\cdot3\cdot4\cdot5} - \frac{a_1}{2\cdot3\cdot5}\right]x^5 + \cdots$$

$$= a_0\left[1 + \frac{x^5}{2\cdot3} - \frac{x^4}{2\cdot3\cdot4} + \frac{x^5}{2\cdot3\cdot4} + \cdots\right] + a_1$$

$$\left[x - x^2/2 + x^3/2\cdot3 + x^4/2\cdot3\cdot4 + \cdots\right]$$

$$y = a_0y_1(x) + a_1y_2(x)$$

where, $y_1(x) = 1 + x^2/2\cdot3 - x^4/2\cdot3\cdot4 + x^5/2\cdot3\cdot4\cdot5$

$y_2(x) = x - x^2/2 + x^3/2\cdot3 + x^4/2\cdot3\cdot4 + \cdots$

Satisfies the condition $y_1(0) = 1$

$y_1'(0) = 0$, $y_2(0) = 0$ and $y_2''(0) = 1$.