III - Electrodynamics

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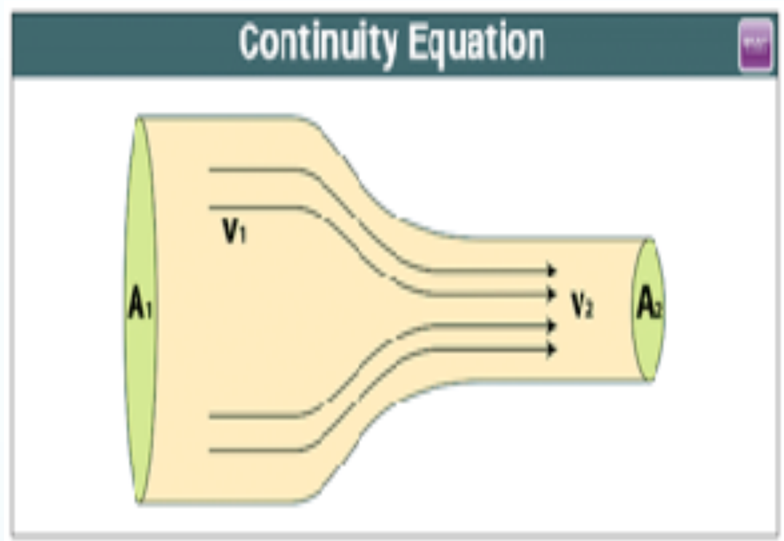
Continuity Equation

The continuity equation describes the transport of some quantities like fluid or gas. The equation explains how a fluid conserves mass in its motion.

Many physical phenomena like energy, mass, momentum, natural quantities and electric charge are conserved using the continuity equations.

This equation provides very useful information about the flow of fluids and its behavior during its flow in a pipe or hose. The hose, a flexible tube, whose diameter decreases along its length has a direct consequence. The volume water flowing through the hose must be equal to the flow rate on the other end.



The flow rate formula.

The Equation of Continuity and can be expressed as:

$$M = \rho_{i1}\, v_{i1}\, A_{i1} + \rho_{i2}\, v_{i2}\, A_{i2} + \ldots.. + \rho_{in}\, v_{in}\, A_{im}$$

$$M = \rho_{o1}\, v_{o1}\, A_{o1} + \rho_{o2}\, v_{o2}\, A_{o2} + \ldots.. + \rho_{on}\, v_{on}\, A_{om} \ldots\ldots\ldots.(1)$$

Where,

M = Mass flow rate, **ρ =** Density, **v =** Speed, **A =** Area

With uniform density, equation (1) can be modified to:

$\mathbf{Q} = v_{i1} A_{i1} + v_{i2} A_{i2} + \ldots + v_{in} A_{im}$

$\mathbf{Q} = v_{o1} A_{o1} + v_{o2} A_{o2} + \ldots + v_{on} A_{om}$……….(2)

Where,

$\mathbf{Q}$ = Flow rate

$\boldsymbol{\rho}_{\mathbf{i1}} = \rho_{i2} .. = \rho_{in} = \rho_{o1} = \rho_{o2} = \ldots = \boldsymbol{\rho}_{\mathbf{om}}$

The **continuity equation** in **fluid dynamics** describes that in any steady state process, the rate at which mass leaves the system is equal to the rate at which mass enters a system.

The differential form of the continuity equation is:

$$\partial \rho / \partial t + \nabla \cdot (\rho u) = 0$$

Where,

t = Time, ρ = Fluid density, u = Flow velocity vector field

The continuity equation is defined as the product of cross-sectional area of the pipe and the velocity of the fluid at any given point along the pipe is constant.

Equation Of Continuity For Time Varying Fields

Statement: Equation of continuity represents the law of conservation of charge. That is the charge flowing out (i.e. current) through a closed surface in some volume is equal to the rate of decrease of charge within the volume :

$$\mathbf{I = -\,dq / dt} \quad \text{--------------------} \ (1)$$

Where,

I is current flowing out through a closed surface in a volume and

-dq/dt is the rate of decrease of charge within the volume.

As $\mathbf{I = \int J.ds}$ and $\mathbf{q = \int \rho\, dv}$

Where J is the Conduction current density and ρ is the Volume charge density. Substituting the value of I and q in equation (1), it will become

$$\mathbf{\int J.ds = -\int d\rho/dt\ dv} \qquad (2)$$

Applying Gauss's Divergence Theorem to L.H.S. of above equation to change surface integral to volume integral,

$\int$[divergence (J)] dV = -$\int$(dρ/dt) dv

As two volume integrals are equal only if their integrands are equal

divergence (J) = – dρ/dt

This is equation of continuity for time varying fields.

Equation of Continuity for Steady Currents:

As **ρ** does not vary with time for steady currents,

that is **dρ/dt = 0**

divergence (J)= 0

The above equation is the equation of continuity for steady currents.

Any continuity equation can be expressed in an **"integral form"** (in terms of a flux integral), which applies to any finite region, or in a **"differential form"** (in terms of the divergence operator) which applies at a point.

Continuity equations underlie more specific transport equations such as the **convection–diffusion** equation, **Boltzmann transport** equation, and **Navier–Stokes** equations.

Continuity equation in Integral form

Mathematically, the integral form of the continuity equation expressing the rate of increase of q within a volume V is:

$$\frac{dq}{dt} + \oint\!\!\!\!\!\!\iint_S j \cdot d\mathbf{S} = 0$$

The integral form of the continuity equation states that:

The amount of q in a region increases when additional q flows inward through the surface of the region, and decreases when it flows outward;

The amount of q in a region increases when new q is created inside the region, and decreases when q is destroyed;

Apart from these two processes, there is *no other way* for the amount of q in a region to change.

Continuity equation in Differential form $$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = \sigma$$

In the case of conserved quantity $$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

Continuity equation in Electromagnetism $$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

Continuity equation in Fluid dynamics $$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho\mathbf{u} = 0$$

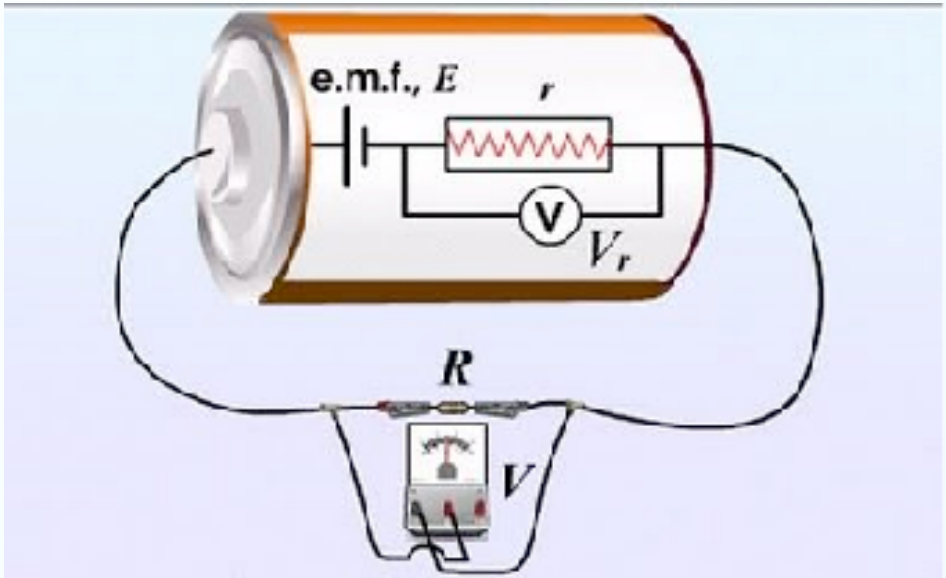
Continuity equation in Energy and heat $$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{q} = 0$$

Continuity equation in Quantum mechanics $$\nabla \cdot \mathbf{j} + \frac{\partial |\Psi|^2}{\partial t} = 0$$

Current Electricity

- Any motion of charges from one section to another section is current.

When two bodies at different potentials are linked with a wire, free electrons stream from B to A, until both the objects reach the same potential, after which the current stops flowing. Until a potential difference is present throughout a conductor, current runs.



- The division of physics that deals with charges in motion is termed as **current electricity.**

The current carried by conductors due to flow of charges is called **conduction current**.

The current due to changing electric field is called **displacement current** or **Maxwell's displacement current**.

Maxwell's displacement current.

Displacement current is a quantity appearing in Maxwell's equations.
Displacement current definition is defined in terms of the rate of change of the electric displacement field (D).
Displacement Current Formula

It can be explained by the phenomenon observed in a capacitor.

Current in a capacitor: When a capacitor start charging, there is no conduction of charge between the plates. However, because of change in charge accumulation with time above the plates, the electric field changes causing the displacement current as below

$$I_D = J_D\, S = S$$

Where,

- **S** is the area of the capacitor plate.
- I_D is the displacement current.
- J_D is the displacement current density.
- **D** is related to electric field E as $D = \varepsilon E$
- **ε** is the permittivity of the medium in between the plates.

Displacement Current Equation

Displacement current has the same unit and effect on the magnetic field as is for conduction current depicted by Maxwell's equation-

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J_D}$$

Where,

H is related to magnetic field B as B = μH

μ is the permeability of the medium in between the plates.

J is the conducting current density.

J_D is the displacement current density.

We know that $\nabla.(\nabla \times \mathbf{H}) = 0$ and $\nabla.\mathbf{J} = -\partial\rho / \partial t = -\nabla.\partial\mathbf{D} / \partial t$
using Gauss's law that is $\nabla.\mathbf{D} = \rho$

Here, **ρ** is the electric charge density.

Thus, $\mathbf{J_D} = \partial\mathbf{D} / \partial t$ displacement current density is necessary to balance RHS with LHS of the equation.

Inconsistency of Ampere's law :

For steady currents,
Ampere's circuit law says

$$\nabla \times H = J$$

Taking divergence on both sides,

$$\nabla \cdot \nabla \times H = \nabla \cdot J$$

Which is the equ of continuity for steady currents
This is inconsistent in the equ of continuity for time varying fields.

To overcome the inconsistency, Maxwell assumed that another term has to be added so that the divergence vanishes.

Making use of Gauss law in the equation of continuity, we have D,

$$\nabla \cdot \mathbf{J} = -\frac{\partial}{\partial t}\nabla \cdot D \qquad (\nabla \cdot D = \rho)$$

=> $$\nabla \cdot \left(\mathbf{J} + \frac{\partial D}{\partial t}\right) = 0$$

$\mathbf{J} + \frac{\partial D}{\partial t}$ - total current density

D - displacement vector

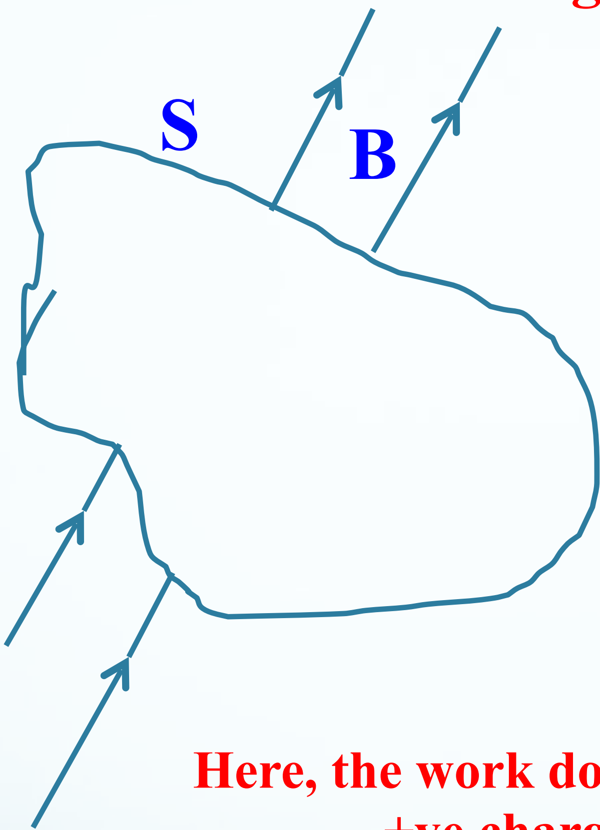
$\frac{\partial D}{\partial t}$ - displacement current density

We now replace J by total current density $\mathbf{J} + \frac{\partial D}{\partial t}$ in Ampere's circuit law,
We have,

$$div\,(\nabla \times H) = div\left(\mathbf{J} + \frac{\partial D}{\partial t}\right)$$

Here $$\nabla \times H = \mathbf{J} + \frac{\partial D}{\partial t}$$ IV-th Maxwells equation

Electromagnetic induction and Faraday's law



From Faraday's law,
we have the induced emf

$$e = -\frac{d\phi}{dt}$$

Let E be the electric field at a point.
Then, the work done in moving a unit +ve charge through dl is E.dl

Here, the work done moving through a unit +ve charge around closed path C is

$$\int E \cdot dl = -\frac{d}{dt}\int B \cdot ds$$

Making use of Stokes theorem, we have

$$\int curl E \cdot ds = -\int \frac{dB}{dt} \cdot ds$$

Maxwell equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \frac{\mathbf{j}}{c^2 \varepsilon_0} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

differential form

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\varepsilon_0}$$ **Gauss's Law for *E***

$$\oint_\Gamma \mathbf{E} \cdot dl = -\frac{\partial}{\partial t}\left(\int \mathbf{B} \cdot d\mathbf{A}\right)$$ **Faraday's Law**

$$\oint_\Gamma \mathbf{B} \cdot dl = \mu_0 \left(I_c + \varepsilon_0 \frac{\partial}{\partial t}\left(\int \mathbf{E} \cdot d\mathbf{A}\right)\right)$$ **Ampere's Law**

$$\oint_S \mathbf{B} \cdot d\mathbf{A} = 0$$ **Gauss's Law for *B***

integral form

$$\mu_0 = \frac{1}{\varepsilon_0 c^2}$$

How are these equivalent?

Maxwell equations and electromagnetic waves

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\varepsilon_0}$$

Gauss's Law for *E*

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\oint_\Gamma \mathbf{E} \cdot dl = -\frac{\partial}{\partial t}\left(\int \mathbf{B} \cdot d\mathbf{A}\right)$$

Faraday's Law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\oint_\Gamma \mathbf{B} \cdot dl = \mu_0 \left(I_c + \varepsilon_0 \frac{\partial}{\partial t}\left(\int \mathbf{E} \cdot d\mathbf{A}\right)\right)$$

Ampere's Law

$$\nabla \cdot \mathbf{B} = 0$$

$$\oint_S \mathbf{B} \cdot d\mathbf{A} = 0$$

Gauss's Law for *B*

differential form | **integral form**

$$\mu_0 = \frac{1}{\varepsilon_0 c^2}$$

Poynting's Theorem

The Poynting Theorem is in the nature of a statement of the conservation of energy for a configuration consisting of electric and magnetic fields acting on charges.

Consider a volume V with a surface S. Then the time rate of change of electromagnetic energy within V plus the net energy flowing out of V through S per unit time is equal to the negative of the total work done on the charges within V.

A second statement can also explain the theorem - "The decrease in the electromagnetic energy per unit time in a certain volume is equal to the sum of work done by the field forces and the net outward flux per unit time".

This is summarized in differential form as:

$$-\frac{\partial u}{\partial t} = \nabla \cdot \mathbf{S} + \mathbf{J} \cdot \mathbf{E}$$

where ∇•S is the divergence of the Poynting vector (energy flow) and J•E is the rate at which the fields do work on a charged object (J is the current density corresponding to the motion of charge, E is the electric field, and • is the dot product).

Using the divergence theorem, Poynting's theorem can be rewritten in **INTEGRAL FORM:**

$$-\frac{\partial}{\partial t}\int_V u\,dV = \oiint_{\partial V} \mathbf{S}\cdot d\mathbf{A} + \int_V \mathbf{J}\cdot\mathbf{E}\,dV$$

where is the boundary of a volume *V*. The shape of the volume is arbitrary but fixed for the calculation.

ELECTRICAL ENGINEERING

In electrical engineering context, the theorem is usually written with the energy density term *u* expanded in the following ways which resembles the continuity equation:

$$\nabla\cdot\mathbf{S} + \epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} + \frac{\mathbf{B}}{\mu_0}\cdot\frac{\partial\mathbf{B}}{\partial t} + \mathbf{J}\cdot\mathbf{E} = 0,$$

where

- ε_0 is the electric constant and μ_0 is the magnetic constant.
- $\epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}$ is the density of reactive power driving the build-up of electric field,
- $\frac{\mathbf{B}}{\mu_0}\cdot\frac{\partial\mathbf{B}}{\partial t}$ is the density of reactive power driving the build-up of magnetic field, and
- $\mathbf{J}\cdot\mathbf{E}$ is the density of electric power dissipated by the Lorentz force acting on charge carriers.

Poynting's Theorem and Vector

As em waves propagate through space, from the source to the receiver, there exists a simple relation between the rate of energy transfer and the amplitude of electric and magnetic fields. This is embodied in "Poyntings Theorem".

Statement :

The vector product of electric (E) and magnetic fields (H) at any point (E X H) gives a measure of rate of energy flow per unit area at that point.

Quantitative Analysis:

Considering the Maxwell's equation associated with Ampere's and Faraday's laws,

$$\text{Curl E} = -\frac{\partial B}{\partial t} \qquad \text{------------------- (1)}$$

$$\text{Curl H} = \mathbf{J} + \frac{\partial D}{\partial t} \qquad \text{------------------- (2)}$$

Taking a scalar product of equ. (2) with E and equ.(1) with –H, we have

$$\mathrm{E} \cdot \mathrm{Curl}\ \mathrm{H} = E \cdot J + E \cdot \frac{\partial D}{\partial t}$$ ------------------- **(3)**

$$\text{-H} \cdot \mathrm{Curl}\ \mathrm{E} = H \cdot \frac{\partial B}{\partial t}$$ ------------------- **(4)**

Adding eqn (3) and eqn (4), we have

$$\mathrm{E} \cdot \mathrm{Curl}\ \mathrm{H}\ \text{-H} \cdot \mathrm{Curl}\ \mathrm{E} = E \cdot J + \left(E \cdot \frac{\partial D}{\partial t} + H \cdot \frac{\partial B}{\partial t} \right)$$

We know that

$$\mathrm{H} \cdot \mathrm{Curl}\ \mathrm{E} - \mathrm{E} \cdot \mathrm{Curl}\ \mathrm{H} = div(E \times H)$$

$$-\ div(E \times H) = J \cdot E + \left(E \cdot \frac{\partial D}{\partial t} + H \cdot \frac{\partial B}{\partial t} \right)$$

Now,

$$E \cdot \frac{\partial D}{\partial t} = \epsilon_0 \, \epsilon_r \, E \cdot \frac{\partial E}{\partial t} = \frac{1}{2} \epsilon_0 \, \epsilon_r \, \frac{\partial}{\partial t}(E \cdot E) = \frac{1}{2} \frac{\partial}{\partial t}(E \cdot D)$$

where, $D = \epsilon E$

Similarly

$$H \cdot \frac{\partial B}{\partial t} = \mu_0 \, \mu_r \, H \cdot \frac{\partial H}{\partial t} = \frac{1}{2} \mu_0 \, \mu_r \, \frac{\partial}{\partial t}(H \cdot H) = \frac{1}{2} \frac{\partial}{\partial t}(H \cdot B)$$

where, $B = \mu H$

Hence, we have

$$J \cdot E + \frac{1}{2} \frac{\partial}{\partial t}(E \cdot D + H \cdot B) + div(E \times H) = 0 \quad \text{---------- (5)}$$

Multiplying the above equation (5) by volume element $d\tau$ and Integrating over volume τ enclosing the surface S

We have,

$$\int_\tau (J\cdot E)d\tau + \int_\tau \frac{1}{2}\frac{\partial}{\partial t}(E\cdot D + H\cdot B)d\tau + \int_\tau div(E\times H)d\tau = 0$$

$$\int_\tau (J\cdot E)d\tau + \frac{1}{2}\int_\tau \frac{\partial}{\partial t}(E\cdot D + H\cdot B)d\tau + \oint_S (E\times H)\cdot dS = 0 \text{---------- (6)}$$

(i) Interpretation of $\int_\tau (J\cdot E)d\tau$

Let the current distribution at the position of charge q_i move with velocity v_i

We have, $\int_\tau (J\cdot E)d\tau = \int_\tau I\,dl\cdot E$ $\quad (J\cdot dl = I\cdot dl)$

$$\int \frac{dq}{dt} \cdot dl \cdot E = \int dq \cdot v \cdot E = \sum_i q_i (v_i \cdot E_i)$$

E_i : electric field at the position of charge q_i

Electromagnetic force acting on the ith charged particle given by Lorentz expression

$$F_i = q_i (E_i + v_i \times B_i)$$

Hence, workdone per unit time on the charge q_i by the field is given by

$$\frac{\partial w_i}{\partial t} = \frac{F \cdot dl}{\partial t} = F \cdot v = F_i \cdot v_i = q_i (E_i + v_i \times B_i) \cdot v_i$$

$$= q_i \cdot (v_i \cdot E_i)$$

Hence, the rate of doing work is given by

$$\frac{\partial w}{\partial t} = \sum_i \frac{dw_i}{\partial t} = \sum_i q_i \cdot (v_i \cdot E_i) = \int_\tau (J \cdot E) d\tau$$

(ii) Interpretation of $\frac{1}{2}\int_\tau \frac{\partial}{\partial t}(E\cdot D + H\cdot B)d\tau$

If we allow the volume τ to be arbitrarily large, the surface integral in equ.(6) can be made to vanish by placing the surface S sufficiently far away from the surface so that the field cannot propagate to this distance in finite time,

(i.e)

$$\oint_S (E\times H)\cdot dS = 0$$

$$\frac{\partial}{\partial t}\int_\tau \frac{1}{2}(E\cdot D + H\cdot B)d\tau + \frac{\partial w}{\partial t} = 0$$

(i.e)

$$\frac{\partial}{\partial t}\left(\int \frac{1}{2}(E\cdot D + H\cdot B) + W\right) = 0 \qquad \text{---------- (7)}$$

Since W involves velocities, it represents K.E of the system.
This means,

$$U = \int_{all\ space} \frac{1}{2}(E\cdot D + H\cdot B)d\tau$$

Equ.(7) represents conservation of energy

(iii) Interpretation of $\oint_S (E \times H) \cdot dS = 0$

For the conservation of energy,

$$\frac{\partial U}{\partial t} + \frac{\partial W}{\partial t} = -\oint_S (E \times H) \cdot dS$$

LHS: describes the rate of change of energy of em field and of the particles contained within the volume τ . Thus, the surface integral must be considered as the energy flowing out of the volume element τ bounded by the surface S per unit time.

The Vector $S = (E \times H)$ **represents the amount of em field energy passing through unit area of the surface in unit time and the direction of flow is normal to vectors E and H**

$S = (E \times H)$ **is called Poynting Vector**

In differential Form, we can write the above equation

$$J \cdot E + \frac{\partial U}{\partial t} + \nabla \cdot S = 0$$

Represents the conservation of energy

Electromagnetic Potentials ($\vec{A}$ and ϕ)

(Scalar and Vector Potentials)

We have from the celebrated Maxwell's equations,

$$\operatorname{div} D = \rho \qquad (1)$$

$$\operatorname{div} B = 0 \qquad (2)$$

$$\operatorname{curl} H = J + \frac{\partial D}{\partial t} \qquad (3)$$

$$\operatorname{curl} E = -\frac{\partial B}{\partial t} \qquad (4)$$

$$\vec{B} = \operatorname{Curl} \vec{A} \qquad (5)$$

This means that vector B can be represented by curl of another vector A

Where $\vec{A}$ is a vector point function

From equation (2), it is obvious that vector B is Solinoidal function of (x,y,z,t)

Substituting equation (5) in equation (4)

We have, $$\operatorname{curl} E = -\frac{\partial}{\partial t}\left(Curl \vec{A}\right)$$

$$\operatorname{curl}\left(E + \frac{\partial A}{\partial t}\right) = 0 \quad (6)$$

From equation (6), it is obvious that $\left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right)$ **is irrotational and must be equal to the gradient of some scalar point function**

$$\overrightarrow{E} + \frac{\partial \overrightarrow{A}}{\partial t} = -\operatorname{grad}\phi \quad (7)$$

$$\overrightarrow{E} = -\operatorname{grad}\phi - \frac{\partial \overrightarrow{A}}{\partial t} \quad (8)$$

Introducing a vector $\overrightarrow{A}$ and a scalar ϕ both being functions of (x, y, z, t) => we have the so called " Electromagnetic potentials")

ϕ **=> Scalar potential**

$\overrightarrow{A}$ **=> Vector potential**

Maxwell's equation in terms of $\overrightarrow{A}$ and ϕ

Considering the fourth Maxwell's equation

$$\mu \cdot \mathrm{Curl}\, H = \mu \cdot J + \mu \cdot \frac{\partial D}{\partial t}$$

$$\mathrm{Curl}\, B = \mu\, J + \mu\, \frac{\partial E}{\partial t}$$

Substituting B and E in terms of $\overrightarrow{A}$ and ϕ, we have

$$\mathrm{Curl}\left(\mathrm{Curl}\,\overrightarrow{A}\right) = \mu\, J + \mu\,\epsilon\,\frac{\partial}{\partial t}\left(-\mathrm{grad}\,\phi - \frac{\partial \overrightarrow{A}}{\partial t}\right)$$

$$\mathrm{grad}\,\mathrm{div}\,\overrightarrow{A} - \nabla^2 \mathrm{A} = \mu\, J - \mu\,\epsilon\,\frac{\partial}{\partial t}\mathrm{grad}\,\phi - \mu\,\epsilon\,\frac{\partial^2 A}{\partial t^2}$$

$$\nabla^2 \mathrm{A} - \mu\,\epsilon\,\frac{\partial^2 A}{\partial t^2} - \mathrm{grad}\left(\mathrm{div}\,\overrightarrow{A} + \mu\,\epsilon\,\frac{\partial\,\phi}{\partial t}\right) = -\mu\, J \quad (9)$$

We know that

$$\mathrm{div}\, D = \rho$$

$$\epsilon\, \mathrm{div}\, E = \rho$$

$$\mathrm{div}\, E = \rho / \epsilon \qquad \Rightarrow \mathrm{div} \left(-\mathrm{grad}\, \phi - \frac{\partial A}{\partial t} \right) = \rho / \epsilon$$

(ie) $$\nabla^2 \phi + \frac{\partial A}{\partial t} (\mathrm{div}\, A) = -\rho / \epsilon$$

Add & Subtract $\mu\, \epsilon \frac{\partial^2 \phi}{\partial t^2}$ **to the above equation**

$$\nabla^2 \phi - \mu\, \epsilon \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial}{\partial t} \left(\mathrm{div}\, A + \mu\, \epsilon \frac{\partial \phi}{\partial t} \right) = -\rho / \epsilon \qquad \text{....(8)}$$

Equations (7) and (8) are Maxwell's equation in terms of $\vec{A}$ **and** ϕ

Nonuniqueness of Electromagnetic potentials:

$$\overrightarrow{B} = \operatorname{curl} \overrightarrow{A}$$

$$\overrightarrow{E} = -\operatorname{grad}\phi - \frac{\partial A}{\partial t}$$

For any $\overrightarrow{A}$ and $\overrightarrow{\phi}$, $\overrightarrow{B}$ and $\overrightarrow{E}$ can be uniquely determined. The Converse is not true. Because Curl gradient of a function vanishes.

We can add gradient of Λ to A without affecting vector B.

$$\overrightarrow{A} = \mathrm{A} + \operatorname{grad}\Lambda$$

$$\overrightarrow{E} = -\operatorname{grad}\phi - \frac{\partial}{\partial t}\left(\overrightarrow{A} - \operatorname{grad}\Lambda\right)$$

$$\overrightarrow{E} = -\operatorname{grad}\left(\phi - \frac{\partial \Lambda}{\partial t}\right) - \frac{\partial \overrightarrow{A}}{\partial t}$$

Replacing Φ by $\phi' = \phi - \frac{\partial \Lambda}{\partial t}$

$$\mathrm{E} = -\mathrm{grad}\,\phi' - \frac{\partial A'}{\partial t}$$

(ie)

$$\mathrm{B} = \mathrm{curl} \quad \mathrm{A}$$

$$\mathrm{B} = \mathrm{curl}\ (A' - \mathrm{grad}\Lambda)$$

$$\mathrm{B} = \mathrm{curl}\, A'$$

$$\mathrm{E} = -\mathrm{grad}\ \left(\phi' + \frac{\partial \Lambda}{\partial t}\right) - \frac{\partial}{\partial t}(A' - \mathrm{grad}\Lambda)$$

$$\mathrm{E} = -\mathrm{grad}\,\phi' - \frac{\partial A'}{\partial t}$$

$$A' = A + \operatorname{grad} \Lambda$$

$$\phi' = \phi - \frac{\partial \Lambda}{\partial t}$$

Λ **=> Gauge function**

$\frac{\partial \Lambda}{\partial t}$ **=> Gauge transformation**

$$B = \operatorname{curl} A$$

$$E = -\operatorname{grad} \phi - \frac{\partial A}{\partial t}$$

Maxwell's equations in terms of $\vec{A}$, ϕ

$$\nabla^2 \mathrm{A} - \mu\,\epsilon \frac{\partial^2 A}{\partial t^2} - \mathrm{grad}\left(\mathrm{div}\, A + \mu\,\epsilon \frac{\partial\,\phi}{\partial t}\right) = -\mu\, J \qquad \text{....(1)}$$

$$\nabla^2 \phi - \mu\,\epsilon \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial}{\partial t}\left(\mathrm{div}\, A + \mu\,\epsilon \frac{\partial\,\phi}{\partial t}\right) = -\rho\,/\epsilon \qquad \text{....(2)}$$

$$\mathrm{div}\, A + \mu\,\epsilon \frac{\partial\,\phi}{\partial t} = 0$$ ➔ **Lorentz Gauge**

$$\nabla^2 \mathrm{A} - \mu\,\epsilon \frac{\partial^2 A}{\partial t^2} = -\mu\, J \qquad \text{....(3)}$$

$$\nabla^2 \phi - \mu\,\epsilon \frac{\partial^2 \phi}{\partial t^2} = -\rho\,/\epsilon \qquad \text{....(4)}$$

$$\Box^2 = \nabla^2 - \mu\,\epsilon \frac{\partial^2}{\partial t^2}$$

$$\Box^2 = \nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}$$

Potentials $\vec{A}$ and ϕ satisfying (3) and (4) Inhomogeneous D'Alembertian equations are called Retarded potentials.

Invariance of Lorentz Gauge

$$\operatorname{div} A + \mu\epsilon \frac{\partial \phi}{\partial t} = 0$$

$$\operatorname{div}\ (\text{A}' - \operatorname{grad}\Lambda) + \mu\epsilon \frac{\partial}{\partial t}\left(\phi' + \frac{\partial \Lambda}{\partial t}\right) = 0$$

$$\operatorname{div} \text{A}' + \mu\epsilon \frac{\partial \phi'}{\partial t} = \nabla^2 \Lambda - \mu\epsilon \frac{\partial^2 \Lambda}{\partial t^2}$$

$$\nabla^2 \Lambda - \mu\epsilon \frac{\partial^2 \Lambda}{\partial t^2} = \Box^2 \Lambda$$

$$\Box^2 \Lambda = 0$$

$$\operatorname{div} \text{A}' + \mu\epsilon \frac{\partial \phi'}{\partial t} = 0$$