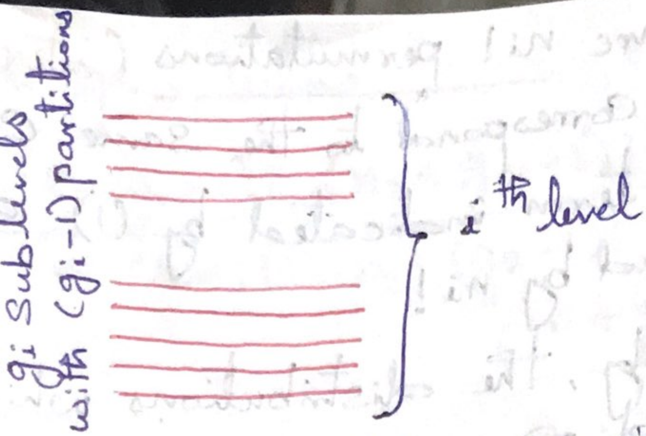


- For this distribution, let us imagine a box divided into g_i sections and the particles be distributed among there sections.
- The choice that which of the compartment will have the sequence (arrange in particular order) can be made in g_i ways.
- Once this has been done, the remaining $(g_i - 1)$ compartments and n_i particles
- (i.e), total particles $(n_i + g_i - 1)$ can be arranged in any order
- (i.e) number of ways of doing this will be equal to $(n_i + g_i - 1)!$
- Thus the total number of ways realize in the distribution will be

$$g_i (n_i + g_i - 1)! \quad \text{—— (1)}$$

- The particles are indistinguishable and therefore rearrangement of particle will not give rise to any distinguishable arrangement.

- There are $n_i!$ permutations (interchange) which correspond to the same configuration hence term indicated by (1) should be divided by $n_i!$

- Secondly, the distributions which can be derived from one another by mere permutation of the cells among themselves does not produce different states, the term (1) should also be divided by $g_i!$

- We thus obtain the required number of ways as

$$\frac{g_i(n_i+g_i-1)!}{g_i!\,n_i!}$$

(or) $$\frac{(n_i+g_i-1)!}{n_i!\,(g_i-1)!}$$

- There will be similar expressions for various other quantum states.
- Therefore, the total number of ways in which n_1 particles can be assigned to the level with the energy ε_1, n_2 to ε_2... and so on is given by the product of such ~~experss~~

expression as given below.

$$G = \frac{(n_1+g_1-1)!}{n_1!\,(g_1-1)!} \cdot \frac{(n_2+g_2-1)!}{n_2!\,(g_2-1)!} \cdots \frac{(n_i+g_i-1)!}{n_i!\,(g_i-1)!}$$

$$= \prod_i \frac{(n_i+g_i-1)!}{n_i!\,(g_i-1)!} \quad \text{--- (2)}$$

- According to the postulates of a prior probabili[ty] ω of eigen states, we have the probability of the system for occuring with the specified distribution to the total number of eigen states (i.e)

$$\omega = \prod_i \frac{(n_i+g_i-1)!}{n_i!\,(g_i-1)!} \times \text{Constant} \quad \text{--- (3)}$$

- So to obtain the contain of maximum probabilit[y] we proceed as follows.

$$\log(x) = x\log(x) - x$$

Taking log of eq (3), we have

$$\log\omega = \sum_i \Big[\log(n_i+g_i-1)! - \log n_i! - \log(g_i-1)!\Big] + \text{Constant}$$

using the stirling's approximation eq (4) becomes

$$\log \omega = \sum_i \Big[(n_i + g_i - 1) \log (n_i + g_i - 1) - (n_i + g_i - 1) - n_i \log n_i + n_i - (g_i - 1) \log (g_i - 1) + (g_i - 1) \Big] + \text{Constant}$$

$$= \sum_i \Big[(n_i + g_i - 1) \log (n_i + g_i - 1) - n_i - g_i + 1 - n_i \log n_i + n_i - (g_i - 1) \log (g_i - 1) + g_i - 1 \Big] + \text{constant}$$

$$= \sum_i \Big[(n_i + g_i - 1) \log (n_i + g_i - 1) - n_i \log n_i - (g_i - 1) \log (g_i - 1) \Big] + \text{Constant.}$$

As compared to n_i & g_i, the value 1 is very small, so we can neglect 1.

$$\log \omega = \sum_i \Big[(n_i + g_i) \log (n_i + g_i) - n_i \log n_i - g_i \log g_i \Big] + \text{Constant.} \quad (5)$$

δ is an partial differentiation symbol.

Differentiate eq (5) with respect to n_i and g_i as Const.

$$\delta \log \omega = \sum_i \delta \left[(n_i+g_i) \log (n_i+g_i) - n_i \log n_i - g_i \log g_i \right]$$

$$= \sum_i \left[\cancel{\delta(n_i+g_i)}^{\delta n_i} \log (n_i+g_i) + \frac{n_i+g_i}{n_i+g_i} \delta n_i - \delta n_i \log n_i - \frac{n_i}{n_i} \delta n_i \right]$$

$$= \sum_i \left[\delta n_i \log (n_i+g_i) - \delta n_i \log n_i \right]$$

$$= -\sum_i \left[\log \frac{n_i}{(n_i+g_i)} \right] \delta n_i \quad \text{--- (6)}$$

The Condition of maximum probability gives

$$\sum_i \left[\log \frac{n_i}{(n_i+g_i)} \right] \delta n_i = 0 \quad \text{--- (7)}$$

the auxiliary Condition to be Satisfied

$$\delta n = \sum \delta n_i = 0 \quad \text{--- (8)}$$

$$\delta E = \sum \varepsilon_i \delta n_i = 0 \quad \text{--- (9)}$$

Applying the Lagrange method of undetermined multipliers (i.e) multiplying eq (8) by α and eq (9) by β and adding the resulting expressions to eq (7) we get,

$$\sum_i \left[\log \frac{n_i}{(n_i+g_i)} + \alpha + \beta \varepsilon_i\right] \delta n_i = 0 \quad \text{—— (10)}$$

As the variation δn_i are independent of each other

$$\log \frac{n_i}{(n_i+g_i)} + \alpha + \beta \varepsilon_i = 0$$

$$1 + \frac{g_i}{n_i} = e^{\alpha + \beta \varepsilon_i}$$

$$\frac{g_i}{n_i} = e^{\alpha + \beta \varepsilon_i} - 1$$

$$n_i = \frac{g_i}{e^{\alpha + \beta \varepsilon_i} - 1}$$

This represents the most probable distributi of the elements among various energy level for a System obeying Bose - Einstein Statisti

Fermi-Dirac Statistics

• In Maxwell-Boltzmann statistics (or) Bose-Einstein statistics, there is no restriction on the particles to be present in any energy state.

• But in case of Fermi-Dirac statistics, applicable to particles like electrons and obeying pauli exclusion principle (no two electrons in an atom have same energy state), only one particle can occupy a single energy state.

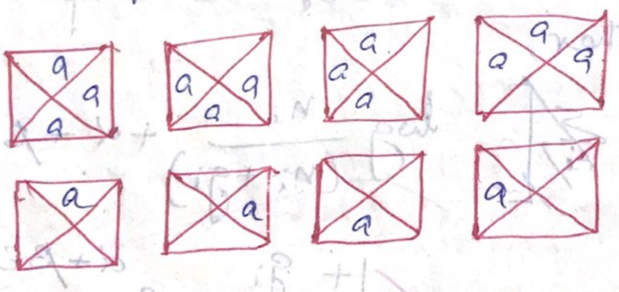
• The distribution of four particles (a, b, c and d) among two cells x and y

• each having 4 energy states.

• Such that there are three particles in cell x while one particles in cell y is shown in fig

cell x — $\omega = 4$

cell y — $\omega = 4$



In this case there will be $4 \times 4 = 16$ possible distribution.

• we consider a general case.

• This statistics is applied to indistinguishable particles having half integeral spin.

• Though the particles are indistinguishable, the restriction imposed (demands cos forces) is that only one particle will be occupied by a single cell.

The Situction of distribution is as follows.

Energy level $\varepsilon_1, \varepsilon_2 \ldots \varepsilon_i \ldots \varepsilon_K$

Degeneracies $g_1, g_2, \ldots g_i \ldots g_K$

Occupation no. $n_1, n_2 \ldots n_i \ldots n_k$

• So in case of fermi dirac statistics, we have the problem of assigning n_i indistinguishable particles to g_i distinguishable levels $\boxed{\text{under the}}$ $\boxed{\text{restriction}}$ that only one particles will be occupied by a single level. $g \geq n_i$

• obviously, clearly g_i must be greater than (or) equal to n_i, because there must be atleast one elementary wavefunction available for every element in the group.

Thus in Fermi-Dirac statistics, the Conditions are,

① . The particles are indistinguishable from each other (i.e) there is no restriction b/w different ways in which n_i particles are choosen.

② . Each sublevel (or) cell may contain 0 (or) one particle. obviously g_i must be greater than (or) equal to n_i.

③ . The Sum of energies of all the particles in the different quantum groups, taken together, constitutes the total energy of the systems.

- Now the distribution of n_i particles among the g_i states can be done in the following way.

- we easily find that the first particle can be put in anyone of the i^{th} level in g_i ways.
- According to pauli exclusion principle no more particles can be assigned to that filled state.
- Thus we are left with $(g_i - 1)$ states in $(g_i - 1)$ ways, and so on. (the 1st particles can be distributed in g_i different ways. lly the second particle can arranged in $(g_i - 1)$ different ways and the process continue)
- Thus the total number of diff ways in which n_i particles can be assigned to g_i states is

$$g_i (g_i - 1)(g_i - 2) \ldots (g_i - n_i + 1),$$

$$= \frac{g_i!}{(g_i - n_i)!} \quad \text{—— (1)}$$

The permutations (interchange) among identical particles do not give distinct distribution, and hence such permutation must be excluded from eq (1) which can be done on dividing it by $n_i!$.

Thus we have the required number as

$$= \frac{g_i!}{n_i!(g_i - n_i)!} \quad \text{—— (2)}$$

The total number of eigen states for whole ~~number~~ system is given by

$$G = \prod_i \frac{g_i!}{n_i!(g_i - n_i)!} \quad \text{—— (3)}$$

The probability of the specific state being proportional to G will be

$$W = \prod_i \frac{g_i}{n_i!\,(g_i - n_i)!} \times \text{Constant} \quad \text{—(4)}$$

To obtain the condition of maximum probability, we proceed as follows:

Taking log of eq (4), we have

$$\log W = \sum_i \left\{ \log g_i! - \log n_i! - \log (g_i - n_i)! \right\} + \text{constant} \quad \text{—(5)}$$

using Stirling approximation, eq (5) reduce to

$$\cancel{\log W = \sum_i \{ g_i \log g_i - n_i \log n_i +}$$

$$\log W = \sum_i \left\{ (n_i - g_i) \log (g_i - n_i) + g_i \log g_i - n_i \log n_i \right\} + \text{Constant} \quad \text{—(6)}$$

Differentiating eq (6) w.r. to n_i, we get

$$\delta \log W = \sum_i \delta \left\{ (n_i - g_i) \log (g_i - n_i) + g_i \log g_i - n_i \log n_i \right\}$$

$$= \sum_i \left\{ \delta n_i \log (g_i - n_i) + \frac{n_i - g_i}{g_i - n_i} (-\delta n_i) - \delta n_i \log n_i - \frac{n_i}{n_i} \delta n_i \right\}$$

$$= \sum_i - \left\{ \log n_i - \log (g_i - n_i) \right\} \delta n_i$$

$$= - \sum_i \left\{ \log \frac{n_i}{(g_i - n_i)} \right\} \delta n_i \quad \text{—(7)}$$

The Condition of maximum probability gives

$$\sum_i \left\{ \log \frac{n_i}{(g_i - n_i)} \right\} \delta n_i = 0 \quad — (8)$$

Introducing the auxiliary Conditions.

$$\delta n = \sum \delta n_i = 0 \quad — (9)$$

$$\delta E = \sum \varepsilon_i \delta n_i = 0 \quad — (10).$$

and applying the Lagrange method of undetermi multipliers (i.e) multiplying eq (9) by α and eq (10) by β and adding the resulting expression to equation (8), we have.

$$\sum_i \left[\log \frac{n_i}{(g_i - n_i)} + \alpha + \beta \varepsilon_i \right] \delta n_i = 0 \quad — (1$$

Since δn_i's can be treated as arbitra

$$\log \frac{n_i}{(g_i - n_i)} = -(\alpha + \beta \varepsilon_i)$$

(or) $$\frac{g_i}{n_i} - 1 = e^{(\alpha + \beta \varepsilon_i)}$$

(or) $$n_i = \frac{g_i}{e^{(\alpha + \beta \varepsilon_i)} + 1} \quad — (12).$$

This is the most probable distribution according to Fermi-Dirac Statistics.

② $\log w = \sum_i \log g_i! - \log n_i - \log(g_i - n_i)!$ + const.

Rearranging the terms

$= \sum -\log(g_i-n_i)! + \log g_i - \log n_i$ + con

By applying stirlings approximation

$= -(g_i - n_i)\log(g_i - n_i) + g_i - n_i)$
$+ g_i \log g_i - g_i - n_i \log n_i$
$+ n_i$ + const

$= (n_i - g_i)\log(g_i - n_i) + \cancel{g_i} - \cancel{n_i} +$
$g_i \log g_i - \cancel{g_i} - n_i \log n_i + \cancel{n_i}$
+ const

$= (n_i - g_i)\log(g_i - n_i) + g_i \log g_i$
$- n_i \log n_i$ + const