UNIT- I VECTOR FIELDS AND VECTOR SPACES

GAUSS DIVERGENCE THEOREM

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Divergence Theorem Statement

The divergence theorem states that the surface integral of the normal component of a vector point function "F" over a closed surface "S" is equal to the volume integral of the divergence of $\vec{F}$ taken over the volume "V" enclosed by the surface S. Thus, the divergence theorem is symbolically denoted as:

$$\iint_V \int \nabla \vec{F} . dV = \iint_S \vec{F} . \vec{n} . dS$$

Divergence Theorem Proof

The divergence theorem-proof is given as follows:

Assume that "S" be a closed surface and any line drawn parallel to coordinate axes cut S in almost two points. Let S_1 and S_2 be the surface at the top and bottom of S. These are represented by z=f(x,y)and z=ϕ(x,y) respectively.

$\vec{F} = F1\vec{i} + F2\vec{j} + F3\vec{k}$, then we have

$$\iiint \partial F3 \partial z dV = \iiint \partial F3 \partial z dxdydz$$

$$\iint R[\int z=f(x,y) z=\Phi(x,y) \partial F3 \partial z]dxdy$$

$$\iint R[F3(x,y,z)]z=f(x,y) z=\Phi(x,y) dxdy$$

$$\iint R[F3(x,y,f)-F3(x,y,\Phi)]dxdy \text{ ——(1)}$$

So, for the upper surface S_2,

$dydx = \cos\gamma_2 dS = \vec{k} . \vec{n_2} dS$

Since the normal vector **n_2** to S_2 makes an acute angle γ_2 with $\vec{k}$ vector,

$dxdy = -\cos\gamma_2 dS_1 = -\vec{k} . \vec{n} . dS_1$

Since the normal vector **n_1** to S_1 makes an obtuse angle γ_1 with $\vec{k}$ vector, then

$\iint_R F_3(x,y,z) dxdy = \iint_{s_2} F_3 \vec{k} . \vec{n_2} dS_2$ —-(2)

$\iint_R F_3(x,y,\Phi) dxdy = \iint_{s_1} F_3 \vec{k} . \vec{n_1} dS_1$ —-(3)

Now, the expression (1) can be written as:

$\iint_R F_3(x,y,z) dxdy - \iint_R F_3(x,y,\Phi) dxdy$ —-(4)

Now, substitute (2) and (3) in (4)

$\iint_{s_2} F_3 \vec{k} . \vec{n_2} dS_2 - \iint_{s_1} F_3 \vec{k} . \vec{n_1} dS_1$

Thus, the above expression can be written as,

$\iint_s F_3 \vec{k} . \vec{n} dS$

Similarly, projecting the surface S on the coordinate plane, we get

$\iiint \frac{\partial F_3}{\partial z} dV = \iint F_3 \vec{k} . \vec{n} dS$

$\iiint \frac{\partial F_2}{\partial y} dV = \iint F_2 \vec{j} . \vec{n} dS$

$\iiint \frac{\partial F_1}{\partial x} dV = \iint F_1 \vec{i} . \vec{n} dS$

Now, add the above all three equations, we get:

$\iiint_v \left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right] dV = \iint_s [F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}] . \vec{n} . dS$

Thus, the divergence theorem can be written as:

$\iiint_v \nabla \vec{F} . dV = \iint_s \vec{F} . \vec{n} . dS$

Hence, proved.

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