UNIT- I VECTOR FIELDS AND VECTOR SPACES

GREEN'S THEOREM

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Green's Theorem Statement

Let C be the positively oriented, smooth, and simple closed curve in a plane, and D be the region bounded by the C. If L and M are the functions of (x, y) defined on the open region, containing D and have continuous partial derivatives, then the Green's theorem is stated as

$$\oint_C (Ldx+Mdy)=\iint_D \left(\frac{\partial M}{\partial x}-\frac{\partial L}{\partial x}\right)dxdy$$

Where the path integral is traversed counterclockwise along with C.

Green's Theorem Proof

The proof of Green's theorem is given here. As per the statement, L and M are the functions of (x,y) defined on the open region, containing D and have continuous partial derivatives. So based on this we need to prove:

To prove: $\oint_C (Ldx + Mdy) = \iint_D (\frac{\partial M}{\partial x} - \frac{\partial L}{\partial x})dxdy$

Proof:

From the given diagram, we get

$\oint_c Ldx = \iint_D (-\frac{\partial L}{\partial y})dA$ .....(1)

and

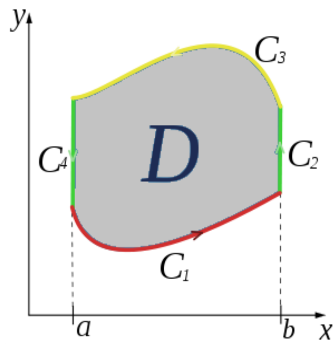
$\oint_c Mdy = \iint_D (\frac{\partial M}{\partial x})dA$ ....(2)

Here, the green's theorem is proved in the first case.

The given diagram has the D region

D = {(x,y) | a ≤ x ≤ b, g1(x) ≤ y ≤ g2(x)}

Here, g1 and g2 are continuous functions on [a, b].



Now, calculate the double integral in (1)

$$\iint_D \frac{\partial L}{\partial y}\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial L}{\partial y}(x,y)\, dy\, dx$$
$$= \int_a^b \Big\{ L(x, g_2(x)) - L(x, g_1(x)) \Big\}\, dx.$$

Now, calculate the line integral (I). From the diagram, C is written as C_1, C_2, C_3, C_4.

With C_1,

$\int_{c_1} L(x,y)dx = \int_a^b L(x, g1(x))dx$ ....(3)

With C_3,

$\int_{c_3} L(x,y)dx = -\int_{-c_3} L(x,y)dx$
$= = -\int_a^b L(x, g2(x))dx$

Therefore, C_3 goes in the negative direction from b to a

Now, C_2 and C_4

$$\int_{C_4} L(x,y)\, dx = \int_{C_2} L(x,y)\, dx = 0.$$

Therefore,

$\int_C L dx = \int_{c_1} L(x,y)dx + \int_{c_2} L(x,y)dx + \int_{c_3} L(x,y)dx + \int_{c_4} L(x,y)dx$

Therefore, the above expression is equal to

$$= \int_a^b L(x, g_1(x))\, dx - \int_a^b L(x, g_2(x))\, dx.$$

Therefore, by combining (3) and (4), we get (1)

Hence, Proved.