ORTHOGONAL CURVILINEAR COORDINATES

An orthogonal coordinate system is a system of curvilinear coordinates in which each family of surfaces intersects the others at right angles. Orthogonal coordinates therefore satisfy the additional constraint that

$$\hat{\mathbf{u}}_i \cdot \hat{\mathbf{u}}_j \equiv \delta_{i\,j},$$

where $\delta_{i\,j}$ is the Kronecker delta. Therefore, the line element becomes

$$\begin{aligned} d\,\mathbf{s}^2 &= d\,\mathbf{r}\cdot d\,\mathbf{r} \\ &= h_1^2\,d\,u_1^2 + h_2^2\,d\,u_2^2 + h_3^2\,d\,u_3^2 \end{aligned}$$

and the volume element becomes

$$\begin{aligned} d\,V &= \left|\left(h_1\,\hat{\mathbf{u}}_1\,d\,u_1\right)\cdot\left(h_2\,\hat{\mathbf{u}}_2\,d\,u_2\right)\times\left(h_3\,\hat{\mathbf{u}}_3\,d\,u_3\right)\right| \\ &= h_1\,h_2\,h_3\,d\,u_1\,d\,u_2\,d\,u_3 \\ &= \left|\frac{\partial r}{\partial u_1}\cdot\frac{\partial r}{\partial u_2}\times\frac{\partial r}{\partial u_3}\right| d\,u_1\,d\,u_2\,d\,u_3 \\ &= \begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial x}{\partial u_2} & \frac{\partial x}{\partial u_3} \\ \frac{\partial y}{\partial u_1} & \frac{\partial y}{\partial u_2} & \frac{\partial y}{\partial u_3} \\ \frac{\partial z}{\partial u_1} & \frac{\partial z}{\partial u_2} & \frac{\partial z}{\partial u_3} \end{vmatrix} d\,u_1\,d\,u_2\,d\,u_3 \\ &= \left|\frac{\partial(x,\,y,\,z)}{\partial(u_1,\,u_2,\,u_3)}\right| d\,u_1\,d\,u_2\,d\,u_3, \end{aligned}$$

where the latter is the Jacobian.

The gradient of a function ϕ is given in orthogonal curvilinear coordinates by

$$\begin{aligned} \text{grad}\,(\phi) &\equiv \nabla\phi \\ &= \frac{1}{h_1}\frac{\partial\phi}{\partial u_1}\hat{\mathbf{u}}_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial u_2}\hat{\mathbf{u}}_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial u_3}\hat{\mathbf{u}}_3, \end{aligned}$$

the divergence is

$$\text{div}\,(F) \equiv \nabla\cdot\mathbf{F} \equiv \frac{1}{h_1\,h_2\,h_3}\left[\frac{\partial}{\partial u_1}(h_2\,h_3\,F_1) + \frac{\partial}{\partial u_2}(h_3\,h_1\,F_2) + \frac{\partial}{\partial u_3}(h_1\,h_2\,F_3)\right],$$

and the curl is

$$\nabla \times \mathbf{F} \equiv \frac{1}{h_1\, h_2\, h_3} \begin{vmatrix} h_1\, \hat{\mathbf{u}}_1 & h_2\, \hat{\mathbf{u}}_2 & h_3\, \hat{\mathbf{u}}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1\, F_1 & h_2\, F_2 & h_3\, F_3 \end{vmatrix}$$

$$= \frac{1}{h_2\, h_3} \left[\frac{\partial}{\partial u_2}(h_3\, F_3) - \frac{\partial}{\partial u_3}(h_2\, F_2) \right] \hat{\mathbf{u}}_1 + \frac{1}{h_1\, h_3} \left[\frac{\partial}{\partial u_3}(h_1\, F_1) - \frac{\partial}{\partial u_1}(h_3\, F_3) \right] \hat{\mathbf{u}}_2 + \frac{1}{h_1\, h_2} \left[\frac{\partial}{\partial u_1}(h_2\, F_2) - \frac{\partial}{\partial u_2}(h_1\, F_1) \right] \hat{\mathbf{u}}_3.$$