Schmidt's Orthogonalisation Process

Introduction

- It is a method for ortho-normalising a set of vectors in an inner product space
- The process takes a finite, linearly independent set

$$S = \{v_1, \dots, v_k\} \text{ for } k \leq n$$

- Generates an orthogonal set $S' = \{u_1, \dots, u_k\}$ that spans the same *k*-dimensional subspace of $\boldsymbol{R}^n$ as S.
- This method is named after Jorgen Pedersen Gram and Erhard Schmidt.
- In the theory of Lie group decompositions it is generalized by the Iwasawa decomposition.

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Introduction

- In mathematics, an orthogonal basis for an inner product space V is a basis for V whose vectors are mutually orthogonal.
- If the vectors of anorthogonal basis are normalized, the resulting basis is an orthonormal basis

Steps

- This process consists of steps that describes how to obtain an orthonormal basis for any finite dimensional inner products.
- Let V be any nonzero finite dimensional inner product space and suppose that {u1, u2, . . . , un} is any basis for V.
- We will form an orthogonal basis from this basis say {v1, v2, . . . , vn}

Steps

- Step 1: Let $v1 = u1$
- Step 2: Let $v_2 = u_2 - proj_{w_1}u_2 = u_2 - \frac{<u_1,v_1>}{\|v_1 1\|^2} v_2$ where W_1 is the space spanned by v_1, and $proj_{w_1}u_2$ is the orthogonal projection of u_2 on W_1.
- Step 3: Let $v_3 = u_3 - proj_{w_2}u_3 = u_3 - \frac{<u_3,v_1>}{\|v_1 1\|^2} v_1 - \frac{<u_3,v_2>}{\|v_2 1\|^2} v_2$ where W_2 is the space spanned by v_1 and v_2.
- Step 4: Let $v_4 = u_4 - proj_{w_2}u_4 = u_4 - \frac{<u_4,v_1>}{\|v_1 1\|^2} v_1 - \frac{<u_4,v_2>}{\|v_2 1\|^2} v_2 - \frac{<u_4,v_3>}{\|v_3 1\|^2} v_2$ where W_2 is the space spanned by v_1 and v_2.

Example

- Let $V = R^3$ with the Euclidean inner product. We will apply the Gram-Schmidt algorithm to orthogonalize the basis $\{(1,-1,1),(1,0,1),(1,1,2)\}$
- Let $u_1 = (1,-1,1)$ $u_2 = (1,0,1)$ $u_3 = (1,1,2)$
- Following the steps:-
- Step 1: Let $u_1 = v_1$ $\rightarrow$ $v_1 = (1,-1,1)$
- Step 2: $v_2 = (1,0,1) - \frac{(1,0,1)(1,-1,1)}{\|(1,-1,1)^2\|}(1,-1,1)$

$$= (1,0,1) - \left(\frac{2}{3}\right)(1,-1,1)$$

$$= \left(\frac{1}{3},\frac{2}{3},\frac{1}{3}\right)$$

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Example

- Step 3: $v_3 = (1,1,2) - \frac{(1,1,2)(1,-1,1)}{\|(1,-1,1)^2\|}(1,-1,1) - \frac{(1,1,2)\left(\frac{1}{3},\frac{2}{3},\frac{1}{3}\right)}{\left\|\left(\frac{1}{3},\frac{2}{3},\frac{1}{3}\right)^2\right\|}\left(\frac{1}{3},\frac{2}{3},\frac{1}{3}\right)$

$$= (1,1,2) - \frac{2}{3}(1,-1,1) - \frac{5}{2}\left(\frac{1}{3},\frac{2}{3},\frac{1}{3}\right)$$

$$= \left(-\frac{1}{2}, 0. \frac{1}{2}\right)$$

- Here $v_1 = v_2 = v_3 = (1,-1,1), \left(\frac{1}{3},\frac{2}{3},\frac{1}{3}\right), \left(-\frac{1}{2}, 0. \frac{1}{2}\right)$ respectively forms an orthogonal basis for R^3