I – NUCLEAR PROPERTIES, TWO BODY PROBLEM AND NUCLEAR FORCES

17.3 GENERAL PROPERTIES OF NUCLEUS

Nuclear size. Rutherford's work on the scattering of α-particles showed that the mean radius of an atomic nucleus is of the order of 10^{-14} to 10^{-15} m while that of the atom is about 10^{-10} m. Thus the nucleus is about 10000 times smaller in radius than the atom.

The empirical formula for the nuclear radius is

$$R = r_o A^{1/3}$$

where A is the mass number and $r_o = 1.3 \times 10^{-15}$ m $= 1.3\, fm$. Nuclei are so small that the fermi (fm) is an appropriate unit of length. $1 fm = 10^{-15}$ m. From this formula we find that the radius of the $_6C^{12}$ nucleus is $R \approx (1.3)(12)^{1/3} = 3 fm$. Similarly, the radius of the $_{47}Ag^{107}$ nucleus is 6.2 fm and that of the $_{92}U^{238}$ nucleus is 8.1 fm.

The nuclear radius may be estimated from the scattering of neutrons and electrons by the nucleus, or by analysing the effect of the finite size of the nucleus on nuclear and atomic binding energies.

Fast neutrons of about 100 MeV energy, whose wavelength is small compared to the size of the nucleus, are scattered by nuclear targets. The fraction of neutrons scattered at various angles can be used to deduce the nuclear size. The results of these experiments indicate that the radius of a nucleus is given by $R \approx r_0 A^{1/3}$ where $r_0 \approx 1.3 - 1.4\, fm$. The scattering can be done with proton beams as well. In this case, however, the effects due to Coulomb interaction have to be separated out. The observations are in agreement with the equation $R \approx r_0 A^{1/3}$ with $r_0 \approx 1.3 - 1.4\, fm$.

The scattering of fast electrons of energy as high as 10^4 MeV, with a wavelength of about 0.1 fm, has the advantage that it can directly measure the charge density inside a nucleus. The results of the experiment are in agreement with the equation $R \approx r_0 A^{1/3}$ but with a somewhat smaller value of $r_0 \approx 1.2\, fm$. The slight difference in the value of r_o may be ascribed to the fact that the electron scattering measures the charge density whereas the neutron and proton scattering experiments measure the region of large nuclear potential, which may be expected to be somewhat larger than the size of the nucleus.

EXAMPLE. *The radius of Ho^{165} is 7.731 fermi. Deduce the radius of He^4.*

SOL. Let R_1, A_1 and R_2, A_2 be the radius and mass number of Ho^{165} and He^4 respectively. Then $R_1 = r_0 A_1^{1/3}$ and $R_2 = r_0 A_2^{1/3}$.

$$\therefore \quad \frac{R_1}{R_2} = \frac{A_1^{1/3}}{A_2^{1/3}} \text{ or}$$

$$R_2 = \frac{R_1 A_2^{1/3}}{A_1^{1/3}} = \frac{7.731 \times 4^{1/3}}{(165)^{1/3}} = 2.238\ fm.$$

Nuclear mass. We know that the nucleus consists of protons and neutrons. Then the mass of the nucleus should be

assumed nuclear mass = $Zm_p + Nm_n$.

Here, m_p and m_n are the respective proton and neutron masses and N is the neutron number. Nuclear masses are experimentally measured accurately by mass spectrometers. Measurements by mass spectrometer, however, show that

$$\text{real nuclear mass} < Zm_p + Nm_n.$$

The difference in masses

$$Zm_p + Nm_n - \text{real nuclear mass} = \Delta m$$

is called the *mass defect.*

Nuclear density. The nuclear density ρ_N can be calculated from $\rho_N = \frac{\text{Nuclear mass}}{\text{Nuclear volume}}$.

Nuclear mass = Am_N where A = mass number and m_N = mass of the nucleon = 1.67×10^{-27} kg.

Nuclear volume = $\frac{4}{3}\pi R^3 = \frac{4}{3}\pi(r_0 A^{1/3})^3 = \frac{4}{3}\pi r_0^3 A$

$$\therefore \quad \rho_N = \frac{Am_N}{\frac{4}{3}\pi r_0^3 A} = \frac{m_N}{\frac{4}{3}\pi r_0^3} = \frac{(1.67 \times 10^{-27})}{\frac{4}{3}\pi(1.3 \times 10^{-15})^3}$$

$$= 1.816 \times 10^{17} \text{ kg m}^{-3}.$$

Note the high value of the density of the nucleus. This shows that the nuclear matter is in an extremely compressed state. Certain stars (the "*white dwarfs*") are composed of atoms whose electron shells have collapsed owing to enormous pressure, and the densities of such stars approach that of pure nuclear matter.

Nuclear charge. The charge of the neucleus is due to the protons contained in it. Each proton has a positive charge of 1.6×10^{-19} C. The nuclear charge is Ze where Z is the atomic number of the nucleus. The value of Z is known from X-ray scattering experiments, from the nuclear scattering of α-particles, and from the X-ray spectrum.

Spin angular momentum. Both the proton and neutron, like the electron, have an intrinsic spin. *The spin angular momentum* is computed by $L_s = \sqrt{l(l+1)}\,\hbar$. Here the quantum number l, commonly called the spin, is equal to 1/2. The spin angular momentum, then has a value $L_s = \frac{\sqrt{3}}{2}\hbar$

Resultant angular momentum. In addition to the spin angular momentum, the protons and neutrons in the nucleus have an *orbital angular momentum.* The resultant angular momentum of the nucleus is obtained by adding the spin and orbital angular momenta of all the nucleons within the nucleus. The total angular momentum of a nucleus is given by $L_N = \sqrt{l_N(l_N+1)}\,\hbar$. This total angular momentum is called *nuclear spin.*

Nuclear magnetic dipole moments. We know that the spinning electron has an associated magnetic dipole moment of 1 Bohr magneton. *i.e.,* $\mu_e = \frac{e\hbar}{2m_e}$. Proton has a positive elementary charge and due to its spin, it should have a magnetic dipole moment. According to Dirac's theory, $\mu_N = \frac{e\hbar}{2m_p}$. Here, m_p is the proton mass.

- μ_N is called a *nuclear magneton* and is the unit of nuclear magnetic moment.

- μ_N has a value of 5.050×10^{-27} J/T. Since $m_p = 1836\ m_e$, the nuclear magneton is only 1/1836 of a Bohr magneton. For nucleons, however, measurements give $\mu_p = 2.7925\ \mu_N$ and $\mu_n = -1.9128\ \mu_N$. Physicists have found that it is especially hard to understand how the neutron, which is a neutral particle, can have a magnetic moment.

Electric quadrupole moment. In addition to its magnetic moment, a nucleus may have an electric quadrupole moment. An electric dipole moment is zero for atoms and nuclei in stationary states. This is a consequence of the symmetry of nuclei about the centre of mass. However, this symmetry does not need to be spherical; there is nothing precluding the nucleus from assuming the shape of an ellipsoid of rotation, for instance. Indeed most nuclei do assume approximately such a shape, and the deviation from spherical symmetry is expressed by a quantity called the *electric quadrupole moment.* It is defined as

$$Q = \left(\frac{1}{e}\right) \int (3z^2 - r^2)\, \rho\, d\tau$$

Here, ρ is the charge density in the nucleus.

- Q is actually a measure of the eccentricity of the ellipsoidal nuclear surface. Evidently $Q = 0$ for a spherically symmetric charge distribution. A charge distribution stretched in the z-direction (prolate) will give a positive quadrupole moment, and an oblate distribution will give a negative quadrupole moment (Fig. 17.1).

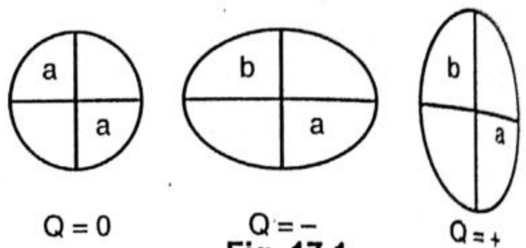


Fig. 17.1

Since the expression is divided by the electronic charge, the dimension of the quadrupole moment is that of an area. In nuclear physics, area is measured in barns. (1 barn = 10^{-28} m^2).

Parity of Nuclei

The total spin of a nucleus consists of the sum of orbital angular momentum of nucleons and the sum of their spins. The orbital angular momentum $\sum_n l_n$ actually defines the parity of the nuclei. The angular momentum eigen function can be expressed as a Spherical Harmonics Y_m which is an even function for even l_n and an odd function for an odd l_n. This shows that $\sum_n l_n$ is either even or odd for nuclei. According to this, the nuclear wave functions are said to have either an even parity or odd parity.

For positive or even parity, $\psi(x, y, z) = \psi(-x, -y, -z)$.

For negative or odd parity, $\psi(x, y, z) = -\psi(-x, -y, -z)$.

If $\sum_n l_n$ is even the parity of nuclide is positive and if $\sum_n l_n$ is odd, the parity is negative.

As an example, the Deuteron nucleus contains a neutron and a proton in S-state with $l = 0$ and the parity of deuteron is positive.

Nuclei of various atoms in a ground state have a definite parity which is either positive or negative. When the nuclei are in the excited state, their parities are not always the same as in the ground state. Parity in nuclear transformations is conservable quantity but is not conserved in weak interactions like Beta decay. It is conserved in nuclear reactions and gamma decay.

Isospin quantum number (T)

Neutrons and protons are similar in all respects except charge. We may regard a *nucleon* as a single entity having two states, the *proton* and the *neutron*. To describe their quantum state, quantum number used is *isospin quantum number* (T). A nucleon is assigned an isospin of 1/2. In an electromagnetic field, two charge states with isospin components of 1/2 and −1/2 can be

3·15. Nuclear Stability, Binding Energy, Mass Defect and Packing Fraction.

On the proton-neutron model of the nucleus, it is clear that the particles which constitute stable nucleus are held together by strong attractive forces and in order to separate them apart, work must be done. In other words, energy must be supplied to the nucleus to separate it into its individual constituents. To consider what form this energy can take place, we apply the famous Einstein's mass energy relation $E = mc^2$, where E and m are the energy and mass of the particle and c is the velocity of light in vacuum. This simply represents that mass and energy are manifestations of the same thing. We should therefore expect the total mass of the nucleus to be less than the sum of masses of the constituents. This has actually been observed through a great deal of experimentation.

The binding energy of the nucleus is defined as the difference between the energy of the constituent particles and the whole nucleus. Let us consider the case of a nucleus ${}_Z M^A$, the binding energy is given by

$$B = [ZM_P + NM_N - {}_Z M^A]\, c^2 \qquad \text{...(3·34)}$$

where M_P = Mass of the Proton,

Z = Number of Protons,

M_N = Mass of the Neutron,

N = Number of Neutrons = $(A - Z)$

${}_Z M^A$ = Measured mass of the neutral atom, [also written as $M(Z, A)$]

The above expression is, now-a-days, generally expressed as

$$B = [ZM_H + NM_N - {}_Z M^A]\, c^2 \qquad \text{...(3·35)}$$

where M_H represents the mass of the neutral hydrogen atom. Since there are A nucleons in the nucleus, the binding energy per nucleon is also given as

$$\frac{B}{A} = \frac{c^2}{A}[ZM_H + NM_N - {}_Z M^A] \qquad \text{...(3·36)}$$

When binding energy fraction B/A is plotted against A, the curve similar to Fig. 3·13 is obtained.

We find from this curve that B/A almost remains constant between $A = 30$ and $A = 100$ and decreases for small and large values of A.

The decrease for large A is due to the coulomb repulsion between the protons which clearly makes the nuclei increasingly less stable. In light nuclei, the individual nucleons are attracted by only a few other nucleons and hence their distances of separation are larger which again reduces the stability. The decrease

Stability of Nucleus and Binding Energy

$$\text{B.E per nucleon} = \frac{\text{Total B.E. of a nucleus}}{\text{The number of nucleons it contains}}.$$

The Binding Energy per nucleon is plotted as a function of mass number A in Fig. 17.3.

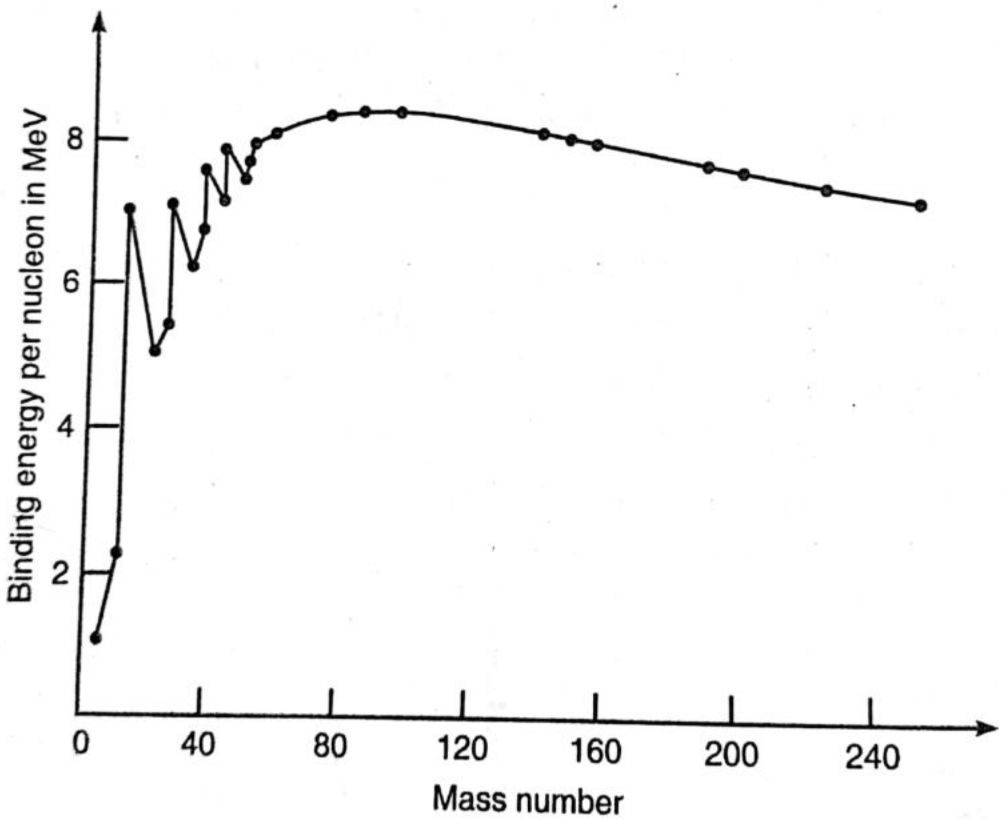


Fig. 17.3

- The curve rises steeply at first and then more gradually until it reaches a maximum of 8.79 MeV at $A = 56$, corresponding to the iron nucleus $_{26}Fe^{56}$.
- The curve then drops slowly to about 7.6 MeV at the highest mass numbers.

Nuclear Fission and Fusion. Evidently, nuclei of intermediate mass are the most stable, since the greatest amount of energy must be supplied to liberate each of their nucleons. This fact suggests that a large amount of energy will be liberated if heavier nuclei can somehow be split into lighter ones or if light nuclei can somehow be joined to form heavier ones. The former process is known as *nuclear fission* and the latter as *nuclear fusion*. Both the processes indeed occur under proper circumstances and do evolve energy as predicted.

Packing fraction. The ratio between the mass defect (Δm) and the mass number (A) is called the packing fraction (f);

$$f = \Delta m / A.$$

Packing fraction means the mass defect per nucleon. Since atomic masses are measured relative to C–12, the packing fraction for this isotope is zero. Packing fraction is a measure of the comparative stability of the atom.

Packing fraction is defined as

$$\text{Packing fraction} = \frac{\textit{Isotopic mass} - \textit{Mass number}}{\textit{Mass number}} \times 10^4.$$

Packing fraction may have a *negative* or a *positive* sign. If packing fraction is negative, the isotopic mass is less than the mass number. In such cases, some mass gets transformed into energy in the formation of that nucleus, in accordance with Einsein's equation $E = mc^2$. Such nuclei, therefore, are more stable. A positive packing fraction would imply a tendency towards instability. But this is not quite correct, especially for elements of low atomic masses.

A plot of packing fraction against the corresponding mass numbers of the various elements is shown in Fig. 17.4. It is seen that helium, carbon and oxygen atoms of mass numbers 4, 12 and 16 respectively, do not fall on this curve. Their paking fractions have small values. These elements are, therefore, stable. The transition elements, with mass numbers in the neighbourhood of 45, have lowest packing fractions with a negative sign, which indicates their high stability. The packing fraction beyond mass number 200 becomes positive and increases with increase in mass number. This indicates increasing instability of these elements. Elements with mass numbers beyond 230 are radioactive and undergo disintegration spontaneously.

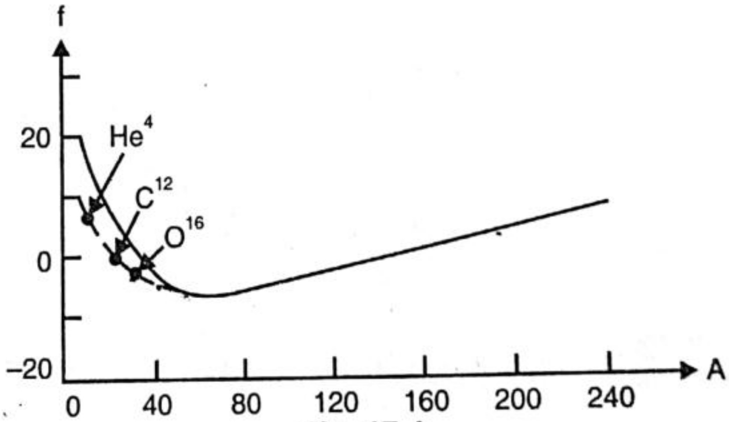


Fig. 17.4

17.5 NUCLEAR STABILITY

Table 17.1 shows how the 272 stable nuclei found in nature are classified according to even and odd numbers of protons and neutrons.

Table 17.1.

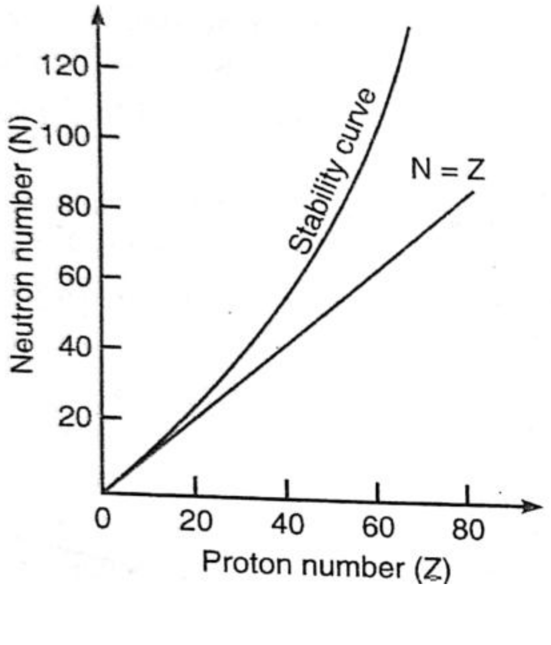
Protons	*Neutrons*	*Stable Nuclides*
even	even	160
even	odd	56
odd	even	52
odd	odd	4
		272

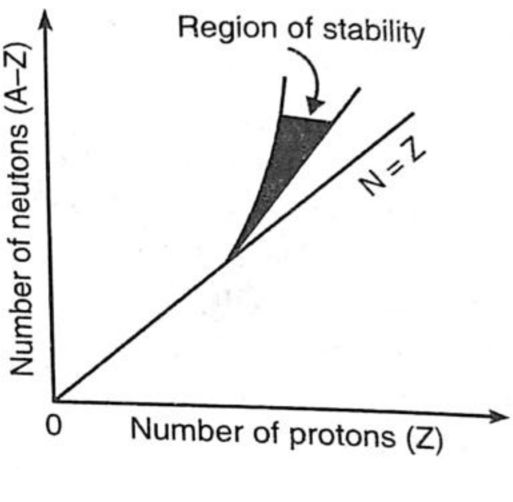
The combination of an even number of protons and an even number of neutrons, composing the nucleus, is evidently preferred by nature for stable nuclides. The odd-odd combination of stable nuclides is found only in the light elements. The number of even-odd combinations is about the same.

A plot of the number of neutrons versus the number of protons for the stable nuclides is shown in Fig. 17.5. Notice that for $Z < 20$, the stability line is a straight line with $Z = N$. For the heavier nuclides $Z > 20$, $N > 20$, the stability curve bends in the direction of $N > Z$. For example ${}_{20}Ca^{48}$ has $N = 28$, $Z = 20$; for larger values of Z, the tendency is more pronounced, as in the case of ${}_{91}Pa^{232}$ which has $N = 141$, $Z = 91$.

Evidently, for large values of Z, the coulomb electrostatic repulsion becomes important, and the number of neutrons must be greater to compensate this repulsive effect.

Thus the curve of Fig. 17.5 departs more and more from the $N = Z$ line as Z increases. For maximum stability, there is an optimum value of neutron/proton ratio. The number of neutrons N $(= A - Z)$ required for maximum stability is plotted as a function of proton number Z in Fig. 17.6. All the stable nuclei fall within the shaded region. Nuclei above and below the shaded region are unstable. Artificial radioactive nuclei lie at the fringe of the region of stability. All nuclei with $Z > 83$, and $A > 209$ spontaneously transform themselves into lighter ones through the emission of α and β particles. α and β decays enable an unstable nucleus to reach a stable configuration.





3.17. Semi-Empirical Mass Formula

We have seen that nuclei can be taken to be spherical with radius $R = R_0 A^{1/3}$. On the basis of this concept and some other classical concepts such as surface tension, electrostatic repulsion etc., a formula for the atomic mass of a nuclide in terms of binding energy correction terms was set up by Weizsacker in 1935 which was later on modified by Bethe and others. This formula can be used to predict the stability of nuclei against particle emission, energy release and stability for fission.

The mass of a nucleus is given by the formula (3·35) is,

$$M_{(Z,A)} = Z M_H + N M_N - B \qquad ..(3\cdot39)$$

where B is the binding energy expressed in mass units. If it was possible to calculate B from a general formula, all nuclear masses could be evaluated theroetically. Weizsacker and others made an attempt in this direction and developed an empirical formula assuming the liquid drop model of the nucleus regarding B as similar to latent energy of condensation. Some of the properties of nuclear forces (saturation, short range etc.) which have been deduced from the approximate linear dependence of the binding energy on the number of particles in the nucleus are analogous to the properties of the forces which hold a liquid drop together. Hence there is ample justification in considering the nucleus to be analogous to a drop of incompressible fluid of very high density 10^{17} Kg/m^3. The value of B was calculated empirically as made up of a number of correction terms given as

$$B = B_1 + B_2 + B_3 + B_4 + \ldots \ldots$$

We shall now proceed to find out empirically the values of B_1, B_2 etc.

Volume Energy Correction

The major contribution to B, *viz.* B_1 comes from the mutual interactions of the nucleons under the influence of nuclear forces. The first correction term expresses the fact that since the nuclei are bound, the nuclear binding energy is proportional to the volume of the nucleus or to the total number of nucleons A and so we can write

$$B_1 \propto v \propto A$$

$$\therefore \quad B_1 = a_1 A \qquad ...(3\cdot40)$$

where a_1 is a positive constant ($a_1 > 0$)

Surface Energy. The above strict proportionality between B_1 and A implicitly assumes overall constancy in the strength of the interaction of each nucleon with its immediate surroundings. However those nucleons which are situated in the surface region of the nucleus are necessarily more weakly bound than those in the nuclear interior because they have fewer immediate neighbours. The number of such nucleons is proportional to the surface area of the nucleus and therfore to R^2 and so is proportional to $A^{2/3}$ (because $R \propto A^{1/3}$). Thus we have

$$B_2 = -a_2 A^{2/3} \quad ...(3 \cdot 41)$$

where, a_2 is a positive constant ($a_2 > 0$).

The sign of surface energy B_2 must be opposite to that of B_1 since this effect which corresponding to the surface tension of a liquid drop, represents a weakening in the binding energy. Carrying this analogy further, we see that this weakening is least and therefore the stability is greatest when the droplet is spherical in shape since then the surface area is minimum for a given volume.

Coulomb Energy

Assuming that the nuclear charge Ze is uniformly distributed throughout the nuclear volume, the coulomb energy of the nucleus has been calculated as

$$E_c = \frac{1}{4\pi\varepsilon_0} \cdot \frac{3}{5} \frac{Z^2 e^2}{R} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{3}{5} \frac{Z^2 e^2}{R_0 A^{1/3}} \quad ...(3 \cdot 42)$$

The assumption that protons are uniformly distributed is far from correct. Moreover protons obey *Pauli's Exclusion Principle* and two of them can not occupy the same place and this effect must be considered in a more accurate determination of E_c. The effect of coulomb self energy on binding energy is diminutive *i.e.*

$$B_3 = -a_3 \frac{Z^2}{A^{1/3}} \quad ...(3 \cdot 43)$$

The negative sign indicates the diminution of energy is due to repulsion effect. Some times and particularly in the case of light nuclei having comparatively small values of Z, the above formula is slightly modified as

$$B_3 = -a_3 \frac{Z(Z-1)}{A^{1/3}}$$

where A_3 is a positive constant ($a_3 > 0$).

Asymmetry Energy

It has been found that for light nuclei, the condition for stability is $N = Z$. This is called *symmetry effect*. Any deviation from $N = Z$, reduces the stability of the nuclei and hence reduces the binding energy. The deficit in binding energy depends on the neutron excess $(N - Z)$ and is proportional to $\frac{(N-Z)^2}{A}$. This symmetry effect is purely a quantum mechanical effect in contrast to the surface energy effect and coulomb energy effect. The correction term is given as

$$B_4 = -a_4 \frac{(N-Z)^2}{A} = -a_4 \frac{(A-2Z)^2}{A} \quad ...(3 \cdot 44)$$

a_4 being a positive constant.

The minus sign represents the weakening in binding caused by asymmetry in N and Z since beyond a certain stage the neutron ceases to act as binding agent within the nucleus.

Pairing Energy

It has been found that the stability of nuclei is very intimately connected with whether the proton number Z or neutron number N, in the nucleus is even or odd. Nuclei with even Z and even N (even-even nuclei) are most stable, even-odd and odd-even nuclei are less stable and odd-odd nuclei are most unstable. To take account of this pairing effect, an additional term is incorporated into the mass formula

$$B_5 = \begin{cases} +\delta & \text{for } e-e \text{ nuclei} \\ 0 & \text{for } e-\text{o nuclei and O}-e \text{ nuclei} \\ -\delta & \text{for o}-\text{o nuclei} \end{cases}$$

Where δ is empirically found to be given by

$$\delta = a_3 A^{-3/4} \qquad \text{...(3·45)}$$

Assembling all the above correction terms, the semi-empirical mass formula from (3·39) is given by

$$M(Z, A) = ZM_H + (A - Z) M_N - a_1 A + a_2 A^{2/3} + a_3 Z^2/A^{1/3} + a_4 \frac{(A - 2Z)^2}{A} \pm \delta \qquad \text{...(3· 46)}$$

where the binding energy correction terms are taken in mass units.

The empirical formula for binding energy is given as

$$B = a_1 A - a_2 A^{2/3} - a_3 \frac{Z^2}{A^{1/3}} - a_4 \frac{(A - 2Z)^2}{A} \pm \delta \qquad \text{...(3· 47)}$$

Dividing this expression (3· 47) by A, we get the binding energy per nucleon. Thus the binding energy per nucleon is given by

$$\frac{B}{A} = a_1 - \frac{a_2}{A^{1/3}} - a_3 . \frac{Z^2}{A^{1/3}} - a_4 . \frac{(A - 2Z)^2}{A^2} \pm \frac{\delta}{A} \qquad \text{...(3· 47a)}$$

The emprical values of the coefficients evaluated by comparison of the above equation with the mass of stable nuclides and energetics of nuclear reactions are listed below—

$a_1 = 14 ·1$ MeV

$a_2 = 13· 0$ MeV

$a_3 = 0·595$ MeV

$a_4 = 19· 0$ MeV

$a_5 = 33·5$ MeV

From equation (3· 46) it is obvious that mass $M(Z, A)$ is a quadratic function of Z for a given mass number A. Thus a graph of $M(Z, A)$ versus Z will be a parabola, the minimum vertex of which will represent the most stable isobar. Experimentally it has been found that for odd-A nuclides, there is only one stable isobar. For even-A nuclides, there are oftern two and sometimes three stable isobars.

It is not possible to find an empirical formula which could fit all A. The formula over a limited range of $A > 15$ is quite successful and could be used to predict the most stable isobars. The accuracy of eq. (3 · 46) is indicated by the following data :

Nucleus	O^{16}	Cr^{52}	Mo^{98}	Au^{197}	U^{238}
Experimental Mass	16·0000	51·956	97·943	197·04	238·12
From mass formula	15·9615	51·959	97·946	197·04	238·12

3·18. Mass Parabolas For Isobaric Nuclei

Isobaric nuclides are characterised by the same mass number A (constant A) but different atomic numbers. To examine their behaviour we write down eq. (3·46) in the abbreviated form as,

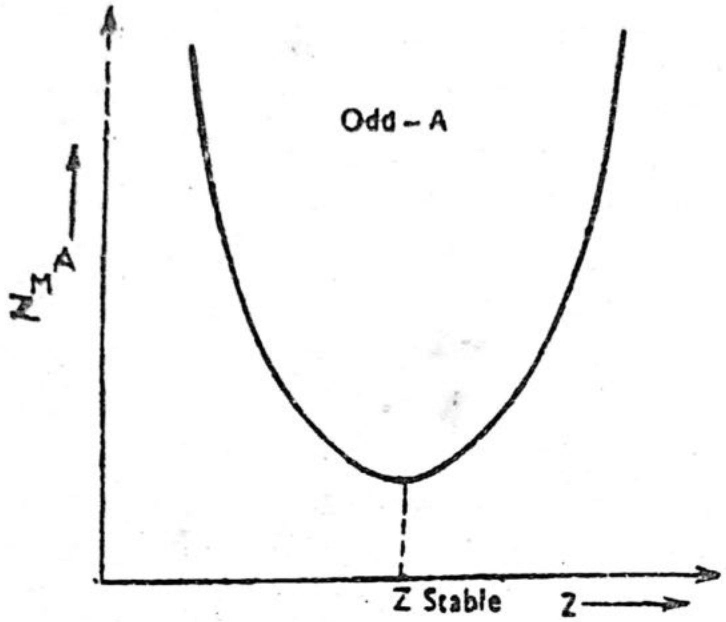


Fig. 3·15. Mass parabola for odd A nuclei. The minima at Z_{stable} respresents the most stable isobar.

$$M(Z, A) \text{ or } {}_ZM^A = K_1 A + K_2 Z + K_3 Z^2 \pm \delta, \qquad ...(3·48)$$

$$\left.\begin{aligned} \text{where} \quad K_1 &= M_N - (a_1 - a_4 - a_2 A^{-1/3}) \\ K_2 &= -[4a_4 + (M_N - M_H)] \\ K_3 &= \left(4\frac{a_4}{A} + \frac{a_3}{A^{1/3}}\right) \end{aligned}\right\} \qquad ...(3·49a)$$

δ in the above expression is independent of A and Z and has zero value for odd-A nuclei.

Thus for odd-A Isobaric Nuclei, we have

$$_ZM^A = K_1 A + K_2 Z + K_3 Z^2 \qquad ...(3·49)$$

For constant A, this equation represents a parabola. If we differentiate equation (3·48), and equate it to zero, the condition for most stable Z is obtained.

$$\left[\frac{\delta}{\delta Z}({}_ZM^A)\right]_A = 0 = K_2 + 2K_3 Z_{stable}$$

$$\therefore \quad Z_{stable} = -\frac{K_2}{2K_3} \qquad ...(3·50)$$

The mass of the stable isobar is, therfore, writen as

$$_{Z_s}M^A = K_1 A + K_2 Z_s + K_3 Z_s^2 \qquad [Z_s = Z_{stable}]$$

$$= K_1 A - 2K_3 Z_s^2 + K_3 Z_s^2 \qquad \text{using eq. (3·50)}$$

$$= K_1 A - K_3 Z_s^2 \qquad ...(3·51)$$

and $$_Z M^A = K_1 A - 2K_3 Z_s Z + K_3 Z^2. \qquad ...(3·52)$$

Subtracting (3·51) from (3·52) we eliminate $K_1 A$ and thus we get

$$_Z M^A - {}_{Z_s} M^A = K_3 (Z^2 - 2 Z_s Z + Z_s{}^2)$$

$$= K_3 (Z - Z_s)^2 \qquad ...(3·53)$$

This gives the parabolic mass relation for **odd - *A* isobaric nuclei** and contains only the coefficient of K_3. **This equation is very much useful in calculating the transition** energies in reactions where Z changes to $Z \pm 1$. (in –ve or +ve β – decays).

Isobaric Nuclei with Even A. The mass-energy profiles for odd-A and even A isobars have a marked distinction. For odd -A, $\delta = 0$ and are therefore represented by a single parabola while for even -A, δ is positive for even-even nuclei and negative for odd-odd nuclei so that the masses of $e - e$ isoars fall on a separate lower parabola than that for the O-O isobars.

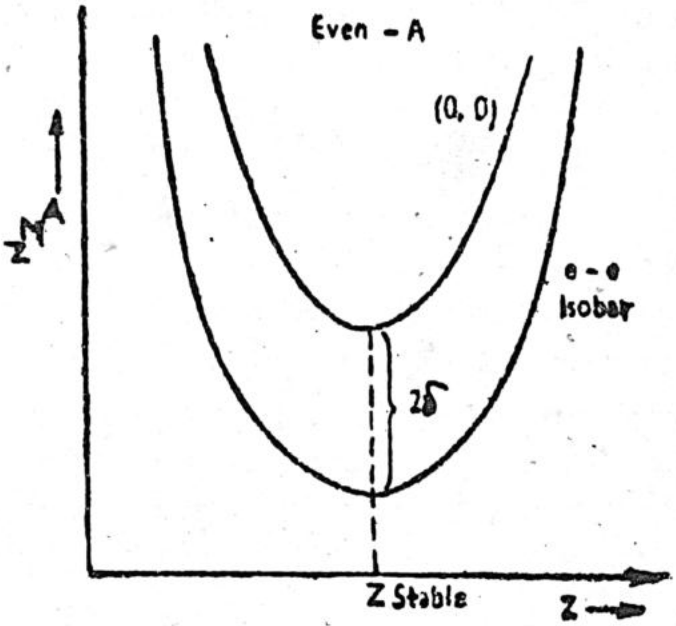


Fig. 3·16. Mass parabolas for even-*A* nuclei having a separation 2 δ where δ is the pairing energy.

Writing equation (3·51) for even-A, odd Z nuclei, we have

$$_{Z_s}M^A = K_1 A - K_3 Z_s^2 + \delta \qquad ...(3·54)$$

and for even-A, even-Z isobars

$$_{Z_s}M^A = K_1 A - K_3 Z_s^2 - \delta \qquad ...(3·55)$$

Thus the vertical separation between the two parabolas is 2 δ .

Semi-empirical Mass Formula and β-Decay Stability.

The β-decay process furnishes an isobaric pair which can be easily studied with the help of semi-empirical mass formula. There are two types of β–decay processes *viz*, β^+ and β^-. In β^-– decay, Z increases by unit and in β^+– decay it decreases by 1 unit while A remains the same. Thus if a nucleus has $Z < Z_s$ the process

$$(A, Z) \rightarrow (A, Z) + \bar{e} + \bar{\nu}$$

is possible if

$$M_{nuc}(A, Z) > M_{nuc}(A, Z + 1) + m_e$$

since the mass of the anti-neutrino is negligibily small. Adding Zm_e to each side of this inequality, the condition may be written in terms of atomic masses

$$M_a(A, Z) > M_a(A, Z+1)$$

Let us consider the following example,

$_{32}Ge^{77}$ decays by a series of β-decays to $_{34}Se^{77}$, Z increasing by 1 at every stage.

$$_{32}Ge^{77} \rightarrow {}_{33}As^{77} + \bar{e} + \bar{\nu} + 2{\cdot}75 \text{ MeV}$$

$$_{33}As^{77} \rightarrow {}_{34}Se^{77} + \bar{e} + \bar{\nu} + 0{\cdot}68 \text{ MeV}$$

$_{34}Se^{77}$ is the only stable nucleus with $A = 77$.

A nucleus with $Z > Z_s$ can decay by emitting β^+ and a neutrino. Let us consider the following example :

$$_{36}Kr^{77} \rightarrow {}_{35}Br^{77} + e^+ + \nu^+ + 2{\cdot}89 \text{ MeV}$$

$$_{35}Br^{77} \rightarrow {}_{34}Se^{77} + e^+ + \nu^+ + 1{\cdot}36 \text{ MeV}$$

For the β^+ decay to be possible, the condition is

$$M_a(A, Z) > M_a(A, Z-1) + 2m_e.$$

Thus odd-*A* nuclei decay to a value of *Z* close to $Z_s\left(= -\dfrac{K_2}{2K_3}\right)$. Since there is only one parabola, there is only one minimum value (Z_s), it is almost impossible to have more than one value of same atomic masses and therefore we expect that for odd-*A* nuclei, there is only one β - stable *Z* value (Fig. 3·10). The following Fig. (3·17) shows that in the case of odd-*A* isobars, only β-decay can take place for the nuclides which lie along the left arm of the parabola and β^+ decay for those lying along the right arm, the decay scheme in each case furnishes with only one isobar.

In the case of nuclei with even *A*, both *Z* and *N* have either even numbers or odd numbers. In the semi-empirical mass formula, δ is positive for *e-e* nuclei and

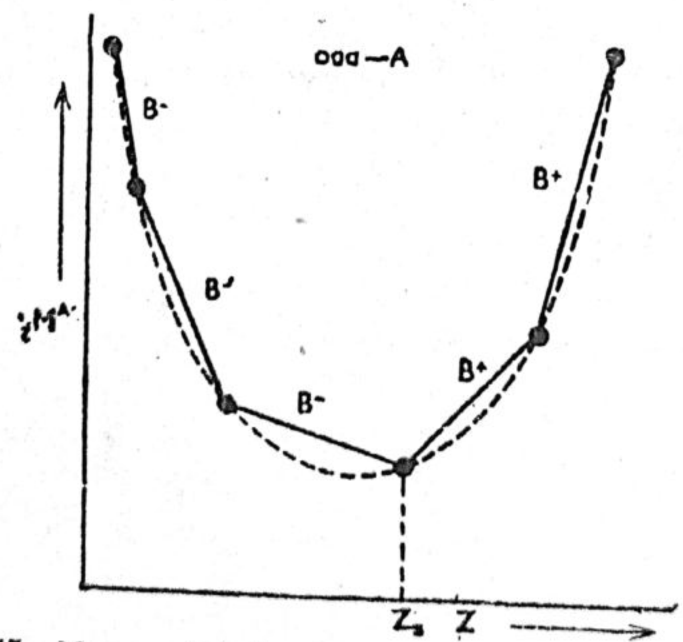


Fig. 3·17. Mass parabola for odd-*A* nuclei showing β^-- decay steps on the left branch and β^+-steps on the right branch, both terminating in a single isobar.

negative for odd-odd nuclei. The *e-e* nuclei have a lower energy than odd-odd nuclei by $2\delta A^{-1/2}$. This quantity varies from 5 MeV for $A = 20$ to 1·4 MeV when $A = 250$. Thus there are two mass parabolas with a relative vertical displacement $2\delta A^{-1/2}/c^2$ as shown in figs. 3·16 and 3·18. The decay chain terminates on the lower of these parabolas because it represents a state of greater stability. The only exceptions to the rule that there are no stable *O-O* nuclei are the light nuclides $H^2, Li^6, B^{10}, N^{14}$ for which the liquid drop is not applicable. However, there is no restriction for the occurence of more than one stable *e-e* isobar for a given mass number *A* since neighbouring isobars on the *e-e* parabola are separated by an interval $\Delta Z = 2$ and can not mormally transmute into each other. Thus it is possible that a single *O-O* parent can have two stable *e-e* daughters— the one formed by β^+ -decay and the other by β^- -decay. A common example is that of Ag^{108} which decays into Cd^{108} by β-emission and stable Pd^{108} by β^+ -emission.

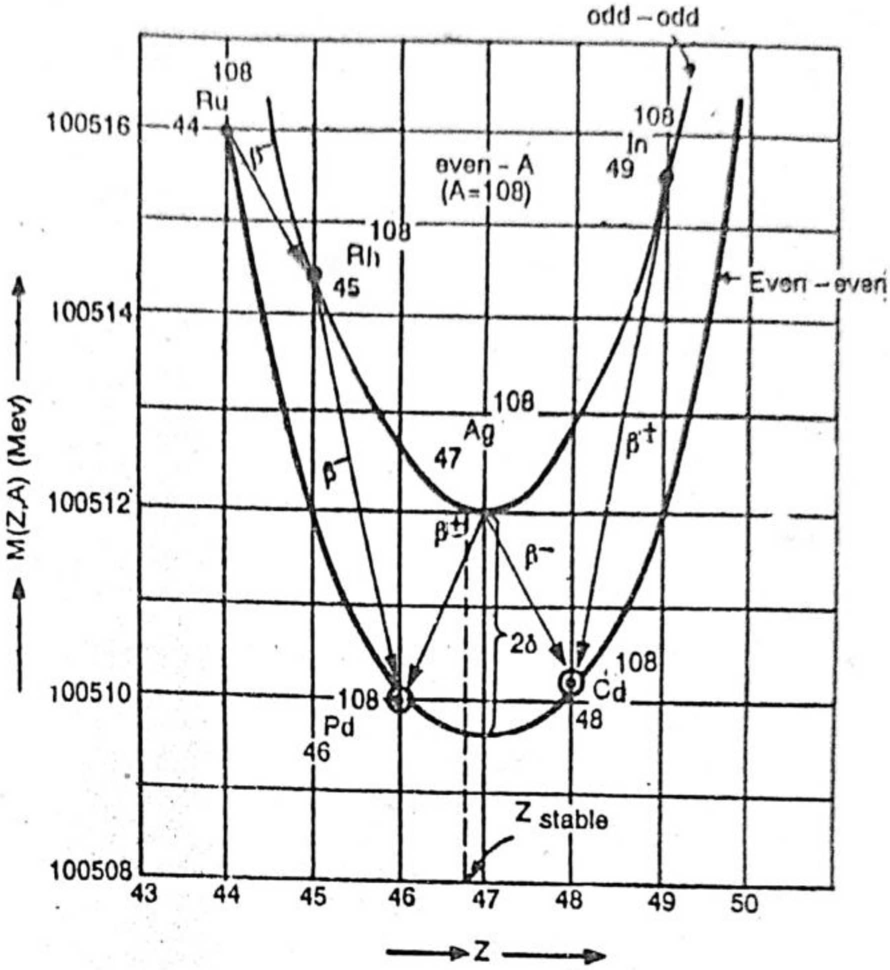


Fig. 3·18. Mass parabolas for even-*A* isobars. *A* = 108, showing two stableisobars Pd and Cd.

The decay energy can be calculated immediately, noting that for β^- -decay, *Z* changes to $(Z + 1)$ and so

$$E_{\beta^-} = {}_Z M^A - {}_{Z+1} M^A = 2 K_3 (Z_s - Z - \tfrac{1}{2}) \qquad ...(3·56)$$

and for β^+ -decay, *Z* changes to $Z - 1$, so that

$$E_{\beta^+} = {}_Z M^A - {}_{Z-1} M^A = -2 K_3 (Z_s - Z + \tfrac{1}{2}) \qquad ...(3·57)$$

Equations (3·56) and (3·57) can be easily combined in following single equation

$$E_{\beta^\pm} = 2 K_3 [\pm (Z - Z_s) - \tfrac{1}{2}] \qquad ...(3·58)$$

The decay energies increase as the separation from the most stable isobar increases. Since the life times are inversely proportional to decay energies, those isobars lying high at the extreme ends of the arms of the parabolas have short life times and consequently the number of observable isobars is restricted.

4·2. Ground State Of Deuteron (Simple Theory)

So far, in the last section, we have put forth the experimental datas for deuteron. In this section our aim is to obtain a theoretical description of deuteron, so that the experimentally observed and theoretically predicted properties could agree. By doing so we hope to get at least some insight into the problem of nature and origin of the nuclear force.

The Schrodinger equation for the two-body problem is

$$\nabla^2\psi(r, \theta, \phi) + \frac{2\mu}{\hbar^2}[E - V(r)]\,\psi(r, \theta, \phi) = 0 \qquad ...(4\cdot 4)$$

where $\mu = \dfrac{m_n m_p}{m_n + m_p} \approx \dfrac{M}{2}$, is the reduced mass of the n-p system

M is the average nucleon mass, $V(r)$ the potential energy taken to be a function of separation between the neutron and the proton and E is the total energy.

In this study of nuclear interactions, the most useful coordinates are a set of spherical polar co-ordinates (r, θ, ϕ) which are connected to the cartesian co-ordinates as follows :

$$\begin{aligned} z &= r\cos\theta \\ y &= r\sin\theta\sin\phi \\ x &= r\sin\theta\cos\phi \end{aligned} \qquad ...(4\cdot 5)$$

where z, is the direction of the polar-axis,

In rectangular coordinates, the Laplacian operator is,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Expressing ∇^2 in terms of the spherical polar coordinates by means of equation (4·5), the Schordinger equation (4· 4) assumes the form :

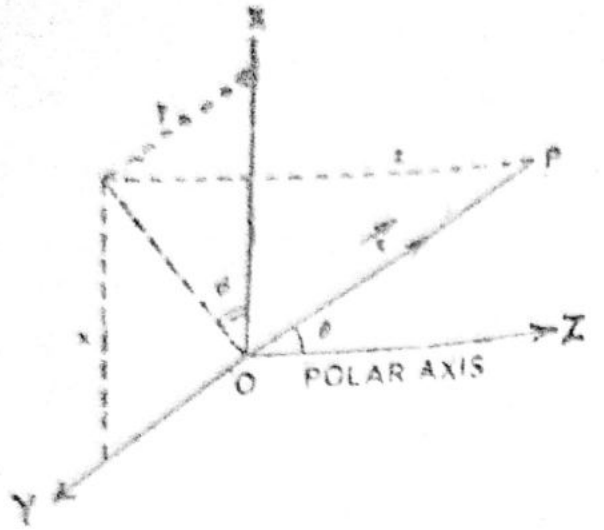


Fig. 4·1

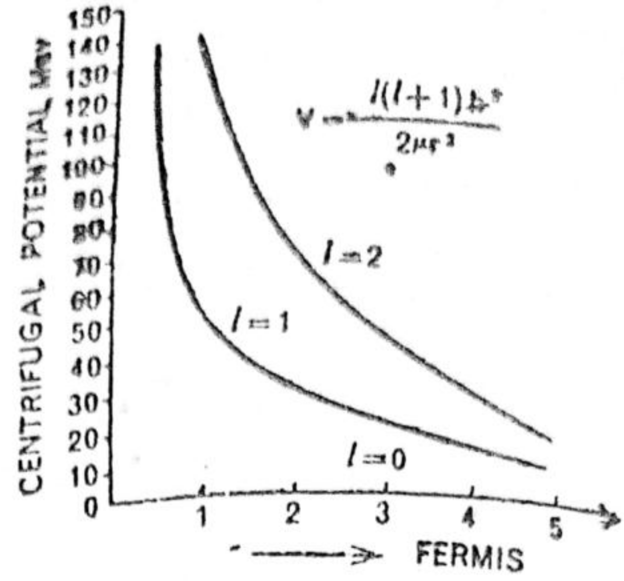


Fig. 4·2 Centrifugal Potential of the neutron proton system

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi + \frac{2\mu}{\hbar^2}[E-V(r)]\,\psi=0 \qquad \text{...(4·6)}$$

If the lowest stable state *i.e.* the ground state of deuteron is taken to be purely a $^3S_1$ state, the force is *central* *i.e.* depends only upon the separation r of the nucleons and not upon the relative velocity or nucleon spin orientation with respect to the line joining the nucleons *i.e.* upon (θ and ϕ). Since central force is a *conservative* one and as such can be expressed as the gradient of a potential function $V(r)$, also spherically symmetrical. This makes the wave function also symmetrical in θ and ϕ. The Schrodinger equation (1·6) when the potential is spherically symmetric, may be separated into angular and radial parts. The radial part of the wave function may be written as

$$\frac{d^2\psi(r)}{dr^2}+\frac{2}{r}\frac{d\psi(r)}{dr}+\frac{2\mu}{\hbar^2}\left[E-V(r)-\frac{l(l+1)\hbar^2}{2\mu r^2}\right]\psi(r)=0 \qquad \text{...(4·7)}$$

The last term *viz.* $\dfrac{l(l+1)\hbar^2}{2\mu r^2}$, is called the *centrifugal potential.*

The centrifugal potential for $l=0$, 1 and 2 is plotted in the fig. (4·2). The centrifugal potential as is seen from the graph, has the effect of tearing the neutron-proton apart. For binding the neutron and proton together as in deuteron, $V(r)$ must be attractive at least over a limited range of r amd should have a value atleast to compensate for the repulsive centifugal potential. This can be most easily achieved in $l=0$ state.

Therefore the lowest quantum mechanical state for a two-body system like deuteron is always an $l=0$ or an S-state.

The Schrodinger equation for an S-state of deuteron ($l = 0$) may therefore be written as

$$\frac{d^2\psi(r)}{dr^2} + \frac{2}{r}\frac{d\psi(r)}{dr} + \frac{2\mu}{\hbar^2}[E - V(r)]\,\psi(r) = 0 \qquad ...(4{\cdot}8)$$

The solution of this equation is greatly simplified by introducing a new wave function $u(r)$, related to $\psi(r)$ as follows :

$$\psi(r) = \frac{u(r)}{r} \qquad ...(4{\cdot}9)$$

Substituting from (4·9) into (4·8), we get

$$\frac{d^2u(r)}{dr^2} + \frac{M}{\hbar^2}[E - V(r)]\,u(r) = 0 \qquad ...(4{\cdot}10)$$

where M = neutron or proton mass and is nearly equal to 2μ.

For the solution of the above equation, we must assume a form for the inter-nucleon potential function $V(r)$. Various potentials can be used subject to the condition that nuclear forces have extremely short range. Some of the commonly used potentials are :

(*a*) **Square-Well Potential ;**

$$V(r) = -V_0 \quad \text{for } r \le r_0$$
$$= 0 \quad \text{for } r > r_0 \qquad ...(4{\cdot}11\,a)$$

where V_0 is the depth of the potential-well and r_0 corresponds to range.

(*b*) **Gaussian Potential :**

$$V(r) = -V_0\, e^{-\left(\frac{r^2}{r_0^2}\right)} \qquad ...(4{\cdot}11b)$$

(*c*) **Exponential Potential :**

$$V(r) = -V_0\, e^{-\left(\frac{r}{r_0}\right)} \qquad ...(4{\cdot}11c)$$

(*d*) **Yukawa Potential :**

$$V(r) = -V_0\,\frac{e^{-(r/r_0)}}{r/r_0} \qquad ...(4{\cdot}11d)$$

Of all these forms, the square-well is the least complicated and can be solved exactly in quantum mechanics. It definitely represents a *short-range* force. In the case of deuteron, fortunately the wave function $u(r)$ is not very sensitive to the exact shape of $V(r)$. Therefore for reasons of simplicity, we use a square-well potential for the solution of deuteron problem.

In equation (4·10) applied to deuteron,

$$E = -E_B = -2{\cdot}226 \text{ MeV}$$

where E_B is positive. Using square well potential, equation (4·10) may be written as

$$\frac{d^2u(r)}{dr^2} + \frac{M}{\hbar^2}(V_0 - E_B)\,u(r) = 0 \quad \text{for } r \le r_0 \qquad ...(4{\cdot}12a)$$

and $$\frac{d^2u(r)}{dr^2} - \frac{M}{\hbar^2}E_B\,u(r) = 0 \quad \text{for } r > r_0 \qquad ...(4\cdot12c)$$

Writing

$$k = \sqrt{\left\{\frac{M}{\hbar^2}(V_0 - E_B)\right\}} \quad \text{and} \quad \gamma = \sqrt{\left(\frac{ME_B}{\hbar^2}\right)} \qquad ...(4\cdot13)$$

equation [4·12 (*a*), (*b*)] may be written as

$$\frac{d^2u(r)}{dr^2} + k^2u(r) = 0 \qquad \text{for } r \le r_0 \qquad ...(4\cdot14\,a)$$

and $$\frac{d^2u(r)}{dr^2} - \gamma^2u(r) = 0 \qquad \text{for } r > r_0 \qquad ...(4\cdot14\,b)$$

The general solution of equation (4·14 *a*) is

$$u(r) = A\sin kr + B\cos kr \qquad ...(4\cdot15\,a)$$

Now $\psi(r)$ must be continuous and bounded and have a continuous derivative everywhere, or in other words $\psi(r)$ must be well behaved (*i.e.* $\psi(r)$ should be finite at $r = 0$ and zero at $r \to \infty$). Therefore $u(r) = r\,\psi(r) \to 0$ as $r \to \infty$, otherwise $\psi(0)$ becomes infinite. This condition on (4·15 *a*) demands that $B = 0$

Hence $\quad u(r) = A\sin kr \qquad r < r_0 \qquad ...(4\cdot15\,b)$

The general solution of equation (4·14 *b*) is

$$u(r) = Ce^{-\gamma r} + De^{\gamma r} \qquad ...(4\cdot16\,a)$$

and the boundary condition at infinity demands that $D = 0$ so that $u(r)$ remains finite. Hence

$$u(r) = Ce^{-\gamma r} \qquad r > r_0 \qquad ...(4\cdot16\,b)$$

Now at $r = r_0$, both $\psi(r)$ [and therefore $u(r)$] and its first derivative must be continuous. Hence from equations (4·15 *b*) and (4·16 *b*) we obtain

$$A\sin kr_0 = Ce^{-\gamma r_0} \qquad ...(4\cdot17a)$$

and $$Ak\cos kr_0 = -C\gamma e^{-\gamma r_0} \qquad ...(4\cdot17b)$$

Dividing equation (4·17*b*) by (4·17*a*), we get

$$k\cot kr_0 = -\gamma \qquad ...(4\cdot18)$$

which does not involve *A* and *C* and relates only the range of the potential r_0 to its depth V_0.

The binding energy of deuteron $E_B = -2\cdot226$ MeV and thus is small which suggests that $E_B << V_0$, may be a resonable assumption. Under this assumption equation (4·18) may be put in a simpler but approximate form. From (4·18), with the help of equation (4·13), we get

$$\cot kr_0 = -\frac{\gamma}{k} = -\frac{\sqrt{(ME_B/\hbar^2)}}{\sqrt{\{M(V_0 - E_B)/\hbar^2\}}}$$

$$= -\sqrt{\left(\frac{E_B}{V_0 - E_B}\right)} = -\sqrt{\left(\frac{E_B}{V_0}\right)} \qquad ...(4\cdot19)$$

where we have neglected E_B in comparision with V_0 in the denominator.

Equation (4·19) suggests that $\cot kr_0$ is small and negative and therefore kr_0 is only slighly greater than $\pi/2$ and not too much error will be involved if we take,

$$k r_0 \approx \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \qquad \text{...(4·20 a)}$$

Under the assumption $E_B << V_0$, equation (4·13) gives

$$k = \sqrt{\left(\frac{MV_0}{\hbar^2}\right)} \qquad \text{...(4·20 b)}$$

Comparing equations [4·20 (a) and (b)], we obtain

$$k = \sqrt{\left(\frac{MV_0}{\hbar^2}\right)} = \frac{\pi}{2r_0}, \frac{3\pi}{2r_0}, \ldots$$

or

$$V_0 r_0^2 = \frac{\pi^2\hbar^2}{4M}, \frac{9\pi^2\hbar^2}{4M}, \ldots \qquad \text{...(4·21 a)}$$

But $kr_0 = 3\pi/2$ is not an acceptable solution. Since $u(r)$ and $\psi(r)$ will then have a node inside the well which means that $\psi(r)$ is not the lowest *i.e.* ground energy level — obviously in ground state, the only acceptable value of kr_0 is $\pi/2$ *i.e.* on the R.H.S. of (4 ·21*a*). Only the first term is to be retained, then

$$V_0 r_0^2 \approx \frac{\pi^2\hbar^2}{4M} \qquad \text{...(4 · 21 b)}$$

This equation gives us only the value of $V_0 r_0^2$ are not the values of V_0 and r_0 separately. From many experimental measurements, it is known that the range of

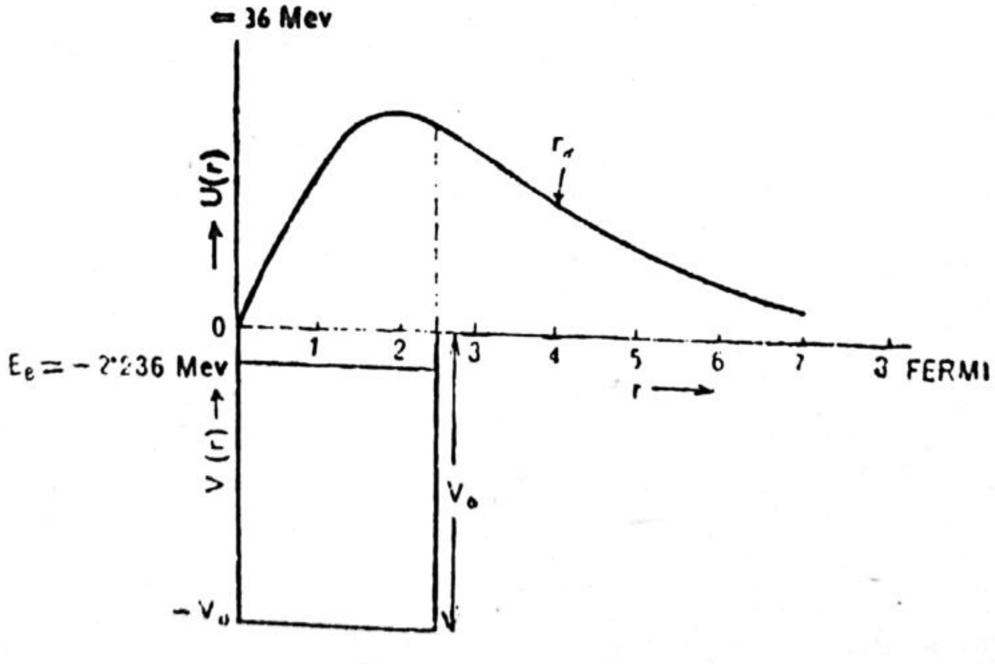


Fig. 4 · 3.

nuclear forces is of the order of 1 Fermi. Accepting the appoximate value of the range $r_0 = 2 \cdot 4$ Fermi, we get

$$V_0 = \frac{\pi^2\hbar^2}{4Mr_0^2} = \frac{(3{\cdot}14)^2(1{\cdot}0545\times10^{-34}\ \text{J-S})^2}{4(1{\cdot}67\times10^{-27}\ \text{Kg})(2{\cdot}4\times10^{-15}\ \text{meter})^2(1{\cdot}6\times10^{-13}\ \text{J})}$$

Since 1 MeV = $1{\cdot}6\times10^{-13}$ Joules

$$= 36\ \text{MeV} \quad \text{...(4·22 a)}$$

This value justifies the assumption that $E_B << V_0$. Fig. (4 · 3) above gives a plot of the ground state wave function, the range of nuclear force and the depth of square-well potential. This potential has a depth of 36 MeV and a width of 2·4 Fermi. As compared with the depth of the potential well *viz.* 36 MeV, the deuteron binding energy $E_B = 2{\cdot}226$ MeV, is small which means that deuteron is just barely bound.

A notable feature of the wave function curve is that the amplitude is appreciably large at distances beyond the range r_0 of the nuclear force. As a matter of fact the exponential decrease of $u(r)$ beyond $r = r_0$ permits us to regard $1/\gamma$ as a measure of the radius r_d of deuteron,

$$r_d = \frac{1}{\gamma} = \frac{\hbar^2}{\sqrt{ME_B}} = \frac{[1{\cdot}0545\times10^{-34}\ \text{J}-\text{S}]}{\sqrt{[1{\cdot}67\times10^{-27}\ \text{Kg}]\times[(2{\cdot}226\times1{\cdot}6\times10^{-13}\ \text{J})]}}$$

$$= 4{\cdot}31\times10^{-15}\ \text{m} = 4{\cdot}31\ \text{Fermi.} \quad \text{...(4·22b)}$$

Thus the radius of deuteron is found to be much larger than the range r_0 of the nuclear force.

Having undertaken the study of the ground state of deuteron, one naturally asks, *at what energies are the other states of the neutron-proton system (deuteron)?* From the study of the ground state, we know that it (ground state) is just barely bound and therefore, it seems quite likely that these states may not exist as bound states for the neutron-proton system. This conculsion has been confirmed in the next section through mathematical analysis of the deuteron problem. But this is not all we want to know about deuteron. We would also like to know about the 1s, $S = 0$ state *i.e.* the state which differs from the ground state only in that the *neutron, proton spins are antiparallel rather than parallel* as considered above. The question, whether this state exists as bound or unbound, is not so pertinent as the one that whether the energy of the state for parallel (↑ ↑) spins of *n*, *p* and that for the antiparallel (↑ ↓) spins, is the same or not ? The study of *n-p* system in the free state (scatteting exeriments) has revealed that this state also does not exist as bound but is unbounded by about 60 K eV.

This means that the energy of the state for parallel (↑ ↑) and anti-parallel spins (↑ ↓) is *different*. From quantum mechanical analysis, we know that the energy of the state depends only upon the potential well, derived from the force. We therefore infer that the nuclear force acting between a neutron and a proton depends upon whether the spins of proton and neutron are parallel or antiparallel (*i.e.* whether total spin *S* of the *n-p* systems is 1 or 0), meaning thereby that it is *spin-dependent* —a notable inference.

The state with neutron-proton spins parallel (↑ ↑) is called a *triplet state* ($^3S_1$) and that with anti-parallel spins (↑ ↓) is called a *singlet state* ($^1S_0$).

4·5. The Meson Theory of Nuclear Forces.

The hypothesis that nuclear forces possess an *exchange character* was put forward by Heisenberg in 1932. However the origin of these exchange forces, at that time, was not known. The only interaction known at that time to change the charge of a nucleon was the β-decay interaction—an exceedingly weak interaction whose strength may be specified by a *dimensionless coupling constant* of the order of 10^{-14}*. However to account for a nuclear binding energy of 8·6 MeV per nucleon, one needs a strong interaction characterised by a coupling constant $g^2/\hbar^2 c$ of the order of unity. No such interaction was known in 1932.

Forces in Physics are derived from quantum field theories. A familiar example of the quantum field theory is the electromagnetic force which is transmitted through the exchange of a *field particle*—the photon, a zero rest mass particle. The electromagnetic field theory accounts for the Coulombian interaction between two charged particles without hypothesizing the philosphically distasteful ideas of action at a distance. It explains as to why this force is propagated at a finite speed (equal to the speed of light) and not instantaneously.

The quantum field theory of the gravitation, in an analogous manner assumes that the gravitational force is propagated by a conjectured field particle—the graviton (yet to be discovered), which like the photon has zero rest mass and travels at a speed equal to that of light. The gravitational force is a result of an exchange of graviton between the interacting masses.

Presumably it was the success of these quantum field theories to explain electromagnetic and gravitational phenomena which inspired H. Yukawa, a Japanese scientist in 1935, to picture the short-range inter-nucleon force in terms of the field quantum, (π -meson) of the nuclear field. The π-mesons were fortunately discovered later in cosmic-ray radiations. The short range of inter-nucleon force assigns a *finite rest mass* to these field quanta, quite unlike the field quanta of the electromagnetic and gravitational fields which have zero rest masses. A considerable body of evidence has established that the range of the nuclear force is about $1{\cdot}4\,F$. When a meson is sent from one nucleon to another, as it must for the propagation of the nuclear force, the creation of the meson violates energy conservation law by an amount ΔE of the order of the meson rest mass $m_{\pi 0}$. *i.e.*

$$\Delta E = m_{0\pi} c^2$$

---

*From Fermi's theory of β decay, a numerical constant characteristic of weak interactions is obtained which has a value $g_F = 1{\cdot}41 \times 10^{-49}$ ergs cm^3 or ($1{\cdot}41 \times 10^{-60}$ Joule-meter^3). To construct a dimensinless coupling contant from this in analogy with gravitational and electromagnetic coupling constant from this in analogy with gravitational and electromagnetic coupling constant, $g_F^{\,2}$ must be divided by a quantity whose dimensions must be $[(\hbar c)^2 (\text{length})^4]$. The fundamental length is taken to be the rationalised compton wave-length of the pion *viz.*

$$\left(\frac{\lambda}{2\pi}\right)_{\pi c} = \frac{\hbar}{m_\pi c} = 1{\cdot}413 \text{ Fermi}$$

Insertion of this value yields a dimensionless *weak-interaction coupling constant* of magnitude:

$$\frac{8F^2}{(\hbar c)^2}\left[\frac{m_\pi c}{\hbar}\right]^4 = 5 \times 10^{-14}$$

The uncertainty relation $\Delta E \cdot \Delta t = \hbar$, fixes a *time-limit* Δt for which such a violation of the energy conservation law can last.

This time limit is given by

$$\Delta t = \frac{\hbar}{\Delta E} = \frac{\hbar}{m_{0\pi} c^2}$$

Even if we assume that the meson travels at the maximum possible speed *viz.*, the speed of light c, the maximum distance it can cover in this time is

$$r_0 = c\,\Delta t = \frac{\hbar}{m_{0\pi}\, c} \qquad \text{...(4·32)}$$

Whence $$m_{0\pi} = \frac{\hbar}{r_{0c}} = \frac{(1 \cdot 0545 \times 10^{-34}\ \text{Joule} \cdot \text{sec})}{(1 \cdot 4 \times 10^{-15}\ \text{m})\,(3 \times 10^{8}\ \text{m/sec})}$$
$$= 0 \cdot 2512 \times 10^{-27}\ \text{Kg}$$

or in terms of electron-mass $m_e = 9{\cdot}1091 \times 10^{-31}$ Kg

$$m_{0\pi} = \frac{0{\cdot}2512 \times 10^{-27}}{9\,{\cdot}1091 \times 10^{-31}}\ \text{electron masses} = 270\ m_e$$

Thus the theoretically predicted mass of the meson ($270\ m_e$) compatible with the short range of the nuclear force ($\cong 1 \cdot 4$ F), is in excellent agreement with the mass of π-meson ($273\ m_e$) measured experimentally in cosmic ray showers. These π-mesons are regarded as the quanta for the nuclear field.

A characteristic difference between nuclear forces and electromagnetic or gravitational forces is that whereas the former have an extremely limited range ($r_0 = 1 \cdot 4\ F$), the latter on account of zero rest mass of their field quanta (photons and gravitons) have an infinite range as can be verified through equation (4·32).

Having known about the properties of the field particle, we now intent to work-out the potential function, known as *Yukawa potential*, responsible for the nuclear forces. We shall present here only a simplified theory.

The relativistic relationship between the total energy E, the momentum p, and the rest mass, m_0, of a particle is

$$E^2 = p^2 c^2 + m_0^2 c^4$$

If in this equation energy E is replaced by the usual quantum mechanical operator $i\hbar\ (\partial/\partial t)$ and momentum p by the quantum mechanical operator $-i\hbar\,\Delta$, we obtain

$$-\hbar^2 \frac{\partial^2}{\partial t^2} = -\hbar^2\, \nabla^2 c^2 + m_0^2\, c^4 \qquad \text{...(4·33)}$$

Let us now introduce a function $\Phi\,(\mathrm{r}, t)$ which has the signficance of potential and which may be regarded as the field variable, then

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \frac{m_0^2\, c^2}{\hbar^2}\, \Phi = 0 \qquad \text{...(4·34)}$$

This is known as the *Klein-Gordon* equation for a spinless relativistic particle. If we set $m_0 = 0$ = rest mass of the photon, then we get

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = 0 \qquad \text{...(4·35)}$$

This is Maxwell's wave equation from which electromagnetic field is derived. This equation may be considered to be derived quantum mechanically from the energy equation for the photons *viz.*,

$$E^2 - p^2c^2 = 0 \qquad \text{...(4·36)}$$

The simplest type of electromagnetic field is the electrostatic field for which $\partial\Phi/\partial t = 0$ and so it obeys the equation

$$\nabla^2\Phi = 0 \qquad \text{...(4·36 a)}$$

known as *Laplace-Equation* which has the well known solution for the potential

$$\Phi = \frac{e}{r} \qquad \text{...(4·37)}$$

For π-mesons, equation (4·34) may be written as

$$\nabla \Phi^2 - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} - \frac{m_{0\pi}^2}{\hbar^2}\Phi = 0 \qquad \text{...(4·38)}$$

The time-independent part of this equation can be obtained by substituting

$$\Phi = \phi(r)\, e^{-i\omega t} \qquad \text{...(4·39)}$$

in equation (4·38) or setting $\partial\Phi/\partial t = 0$ and it is

$$\nabla^2\Phi - \frac{m_{0\pi}^2 c^2}{\hbar^2}\Phi = 0$$

or

$$\nabla^2\Phi - K^2\Phi = 0 \qquad \text{...(4·40 a)}$$

where $K = \frac{m_{0\pi} c}{\hbar} = \frac{1}{r_0}$ from equation (4·32).

Equation (4·40 *a*) valid in case of a meson field, is an analogue of the Laplace's equation valid in case of the electromagnetic field in the absence electric charges. In the presence of *charges*, we require an analogue of *Poission's equation viz.* $\nabla \Phi = 4\pi e \rho$, where e is the electronic charge and ρ is the electron particle density and it is

$$(\nabla^2 - K^2)\Phi = 4\pi g\rho$$

where 'g' is the nucleon *charge* which measures the strength of the interaction between the nucleon and the meson field. If the only nucleon present is a point nucleon fixed at the origin ($r = 0$), $g\rho$ can be replaced by $g\,\delta(r)$, where $\delta(r)$ is the Dirac-delta function

$$[\delta(r) = 1 \text{ at } r = 0 \text{ and } \delta(r) = 0, \text{ at finite } r]$$

then we obtain

$$(\nabla^2 - K^2)\Phi = 4\pi g\,\delta(r) \qquad \text{...(4·40 b)}$$

The solution of equations (4·40 *a*, *b*) as can be seen through direct substitution, is

$$\Phi = -g\frac{e^{-kr}}{r}$$ ...(4·41)

where 'g' is an undermined constant which occupies a place in meson theory analogous to that of the charge e in electromagnetic theory. As we know, in electromagnetic theory, the interaction potential energy of a charge e immersed in an electrostic field Φ is $e\,\Phi$. In the same way it would be expected that in case of a meson field, the interaction potential $V(r)$ of a nucleon of *charge* g with the meson-field Φ should be

$$V(r) = g\,\Phi = -g^2\frac{e^{-kr}}{r}$$

or

$$V(r) = -g^2\frac{e^{-r/r_0}}{r}$$ ...(4·42)

This is then the required Yukawa-potential and is graphically shown in the fig. (4·5).

π-mesons have been discoverd experimentally to be of three kinds, positive (π^+), neutral (π^0) and negative (π^-), all of which are zero spin particles.

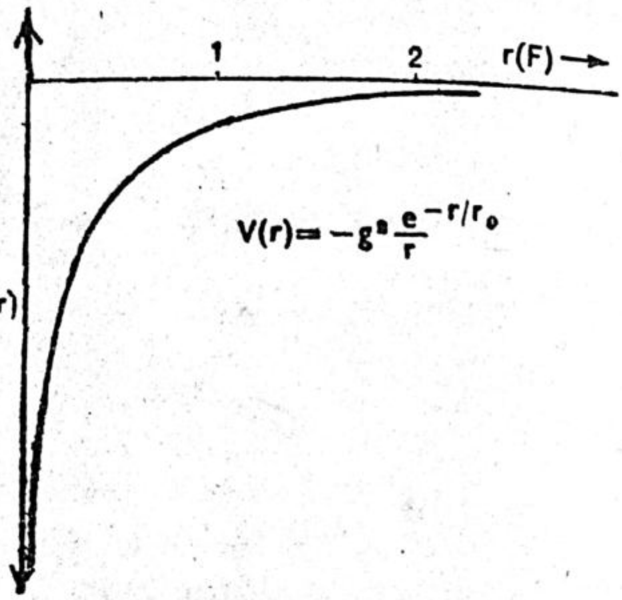


Fig. 4·5. The Yukawa-potential.

However first to be discovered were not the π-mesons but another kind of particle, known as μ-mesons and which were erroneously taken to be the conjectred *Yukawa-particles*, but later they were found to interact very weakly with nuclear matter and moreover they were found to be only of two kinds positive and negative and were $\frac{1}{2}$ spin particles. Thus they could not be the required Yukawa particles as $\frac{1}{2}$ spin particles cannot be transferred from any system without changing its angular momentum. Hence they were discarded as Yukawa particles and the search for the right particle continued until the π-mesons were discovered.

The exchange of π-mesons between the nucleons (protons and neutrons) may be taken to proceed as follows ;

$$n \rightarrow n + \pi^0 \quad \text{and then} \quad p + \pi^0 \rightarrow p$$

4·7. Scattering Cross-Section.

Consider a *thin* rectangular section of a target material having a thickness 'dx' which exposes an area A to a beam of mono-energetic particles impinging upon it. Let the nuclear density of the target *i.e.* the number of target nuclei per unit volume be 'n'. Each target nucleus is supposed to possess an effective area σ, called the

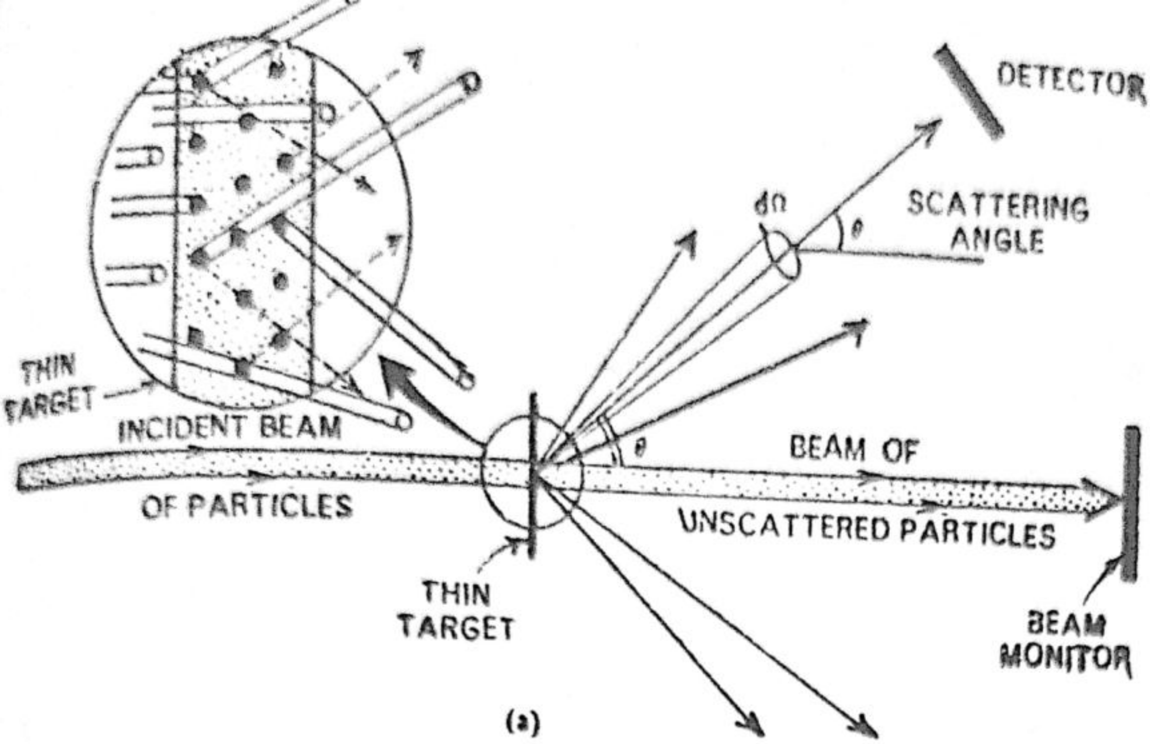


Fig. 4 · 9 (a) A simple scattering experiment

scattering cross-section which in general is very much different from the geometrical cross-section of the nucleus, such that if the incident particle impinges upon the target within this area, a scattering event takes place and if it hits outside this area, it goes undeflected through the target. This area is shown by a small circle in the fig. [4·9 *b*] below.

Cross-section per target nucleus = σ

Total number of target nuclei hit by the incident beam = $n\,A\,dx$

The target is assumed to be thin so that there is no appreciable overlapping or masking of the nuclei of successive layers which might cause a screening of some of the nuclei by those in the preceding layer *i.e.* every target nucleus presents its entire effective cross-section to the beam of incident particles, then

The total target area available for scattering = $(n\,A\,dx)\,\sigma$

The probability for a scattering event then is given by the ratio of the total target area hit by the incident beam. Hence

Probability for scattering

$$P = \frac{\text{Nuclear target area available for scattering}}{\text{Total target area hit by the incident beam}}$$

$$P = \frac{(n\,A\,dx)\,\sigma}{A} = n\,dx\,\sigma \qquad \text{...(4·49)}$$

If N is the total number of particles incident per second at the layer of thickness dx, out of which a number $-$ dN is removed from the indicent beam through scattering, then the same probability may also written as :

$$P = -\frac{dN}{N} = \frac{\text{Number of particles scattered}}{\text{Total number of incident particles}}$$

$\therefore$ from equation (4 · 49)

$$-\frac{dN}{N} = n\,dx\,\sigma \qquad \text{...[4·49 } a]$$

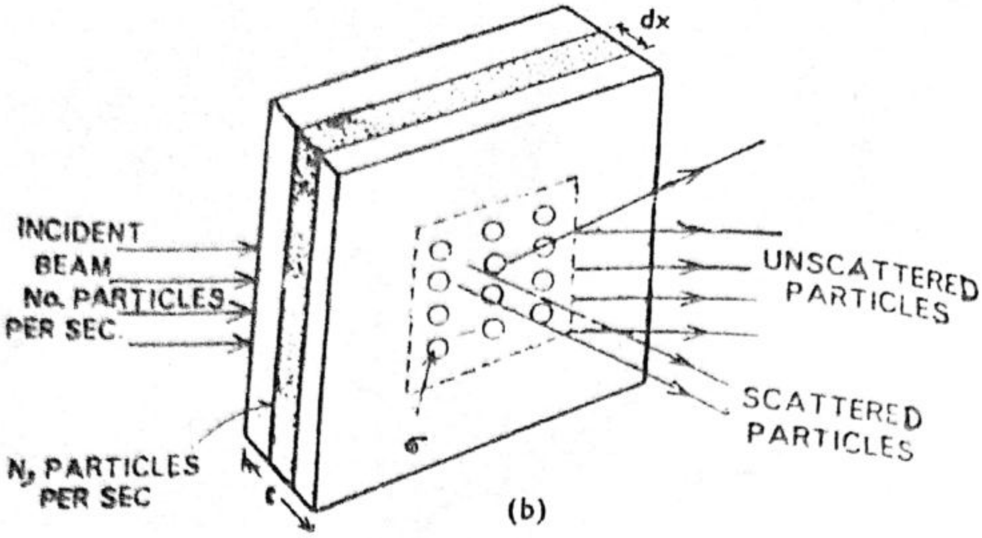


Fig. 4 · 9. (b) Pictorial respresentation of cross-section

Equation [4·49 (*a*)] is valid only for a thin target. However if the target is *thick* this equation is to be integrated over all such sections of thickness dx and then one obtains N_t the number of particles that have traversed the entire thickness 't' of the target material without having been scattered. Thus

$$\int_{N_0}^{N_t}\left(-\frac{dN}{N}\right)=\int_0^t n\sigma\, dx$$

or

$$\log_e \frac{N_t}{N}=-n\sigma t$$

or

$$N_t = N_0\, e^{-n\sigma t} \qquad ...(4·50)$$

Equation (4·50) gives the number of particles passing through the target undeflected. This number is definitely less than the number of particles N_0 incident at $t = 0$. Thus as a result of scattering, the incident beam has become *attenuated*.

The number of particles scattered by the target per second is :

$$N_{sc} = N_0 - N_t = N_0\,(1 - e^{n\sigma t}) \;\cong\; N_0\, n\, \sigma\, t \qquad ...(4·51)$$

if the attenuation is small.

The quantity $n\sigma = \mu$, is called the *attenuation coefficient*. If the target contains a number of isotopes of different cross-sections, $n\sigma$ then is equal to $n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3 + ...$ Equations (4·50) or (4·51) can be written in more useful form by nothing that the number of target *neclei* per unit volume *viz.*, n, is identical to the number of *atoms* per unit volume of the material *i.e.*

$$\mu = n\,\sigma = \frac{\sigma N_A \rho}{A} \qquad ...(4·52)$$

where ρ is the *density* of the target material and A its *atomic weight* and N_A the *Avogadro's number* or *Laschmidt number*.

Therefore

$$N_t = N_0\, e^{-\rho N_A \sigma t/A} \qquad ...(4·50\ a)$$

4·8. Neutron-Proton Scattering At Low Energies

The Neutron-proton scattering cross-section has been measured extensively both at low and high energies of the incident neutrons. In the low energy range most of the measurements of scattering cross-section are due to Melkonian and Rain water *et. al.* A beryllium target bombarded at by deuterons accelerated in a cyclotron, provided the neutron beam which was shot at a target containing free protons. The scattering cross-section as measured by Melkonian is given in fig. (4·10).

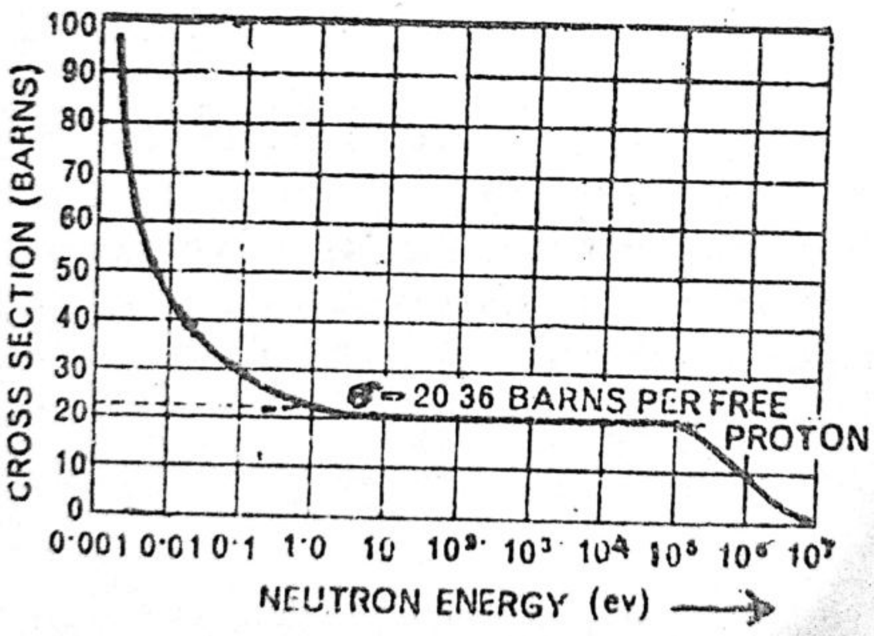


Fig. 4·10. Total *n-p* scattering cross-section at various neutron energies.

These results show that the scattering cross-section depends very much on the energy of the incident neutron. At low energies below 10 MeV, the scattering is essentially due to deuterons having zero angular momentum ($l = 0$) and hence in the centre of mass system, the angular distribution of scattered neutrons is *isotropic*. To account theoretically for the experimental result, we give below a simplified version of the so-called *method of partial waves*. In order to avoid complications due to Coulomb forces we shall consider the scattering of neutrons by *free protons* *viz.* those not bound to molecules. However in practice the protons are of course bound to molecules but the molecular binding energy is only about 0·1 eV. Therefore if the incident neutrons have an energy greater than about 1 eV, the protons can be regarded as free.

In describing elastic scattering events like the scattering of neutrons by free protons, it is more convenient to use the centre of mass system. The quantum mechanical problem describing the interaction between two particles, in the centre of mass system, is equivalent to the problem of interaction between a reduced mass particle and a fixed force centre. For two particles of equal mass such as the (*n-p*) system, the reduced mass is equal to half the mass of either. Although while working out the following theory we shall think in terms of a neutron being scattered by a proton but it applies equally well to spinless, reduced mass particles which are being scattered by a fixed force centre. Let us suppose that the neutron and the proton interact via a spherically symmeteric force field whose potential function is $V(r)$, where r is the distance between the particles.

The Schrodinger equation for a central potential $V(r)$ in the centre of mass system, for the *n-p* system is

$$\left[\nabla^2 + \frac{M}{\hbar}\{E - V(r)\}\right]\psi = 0 \qquad ...(4{\cdot}53)$$

where M is the reduced mass of the *n-p* system .

To analyse the scattering event, we have to solve this equation under proper boundary conditions. Even without actually solving this equation we can tell something about the general character of the solution. In the immediate vicinity of the scattering centre, the action will be violent and its description difficult. At a considerable distance from the scattering centre where the experimentalist lies in wait for the scattered particle, things will however be simpler. For scattering, the boundary condition is that at large distances from the scattering centre the wave should be made up of two parts : (i) an icident plane wave that describes the unscattered particles and superimposed upon it, (ii) an outgoing scattered spherical wave which emanates from the scattering centre. Therrefore we seek a solution of equation (4·53) in the *asymptotic form.* In the light of the above comment,

$$\psi = \psi_{inc} + \psi_{sc} \qquad ...(4{\cdot}54)$$

The wave function that describes an incident plane wave (a beam of particles) moving in the positive Z-direction is

$$\psi_{inc} = e^{ikz} = e^{ikr\cos\theta}$$

where $$k = \sqrt{\left(\frac{ME}{\hbar^2}\right)} \qquad ...(4{\cdot}55)$$

which is a solution of the wave equation (4·53) with $V(r)$ set equal to zero, *viz.* that of the equation whence asymptotically

$$j_l(kr)_{r\to\infty} \to \frac{\sin\left(kr - \frac{l\pi}{2}\right)}{kr} \quad \text{...(4·62)}$$

∴ Asymptotically, $B_l(r)$ from (4·59) are given by

$$B_l(r)_{r\to\infty} \to i^l(2l+1)\frac{\sin\left(kr - \frac{l\pi}{2}\right)}{kr} \quad \text{...(4·59 a)}$$

$$\cong \frac{1}{2ikr} i^l (2l+1)\cdot[e^{i(kr-l\pi/2)} - e^{-i(kr-l\pi/2)}] \quad \text{...(4·59 b)}$$

The Spherical Bessel functions $j_l(kr)$ for various values of l, are given below

$$j_0(kr) = \frac{\sin kr}{kr}$$

$$j_1(kr) = \frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{(kr)}$$

$$j_2(kr) = \left[\frac{3}{(kr)^3} - \frac{1}{kr}\right]\sin(kr) - \frac{3\cos(kr)}{(kr)^2}$$

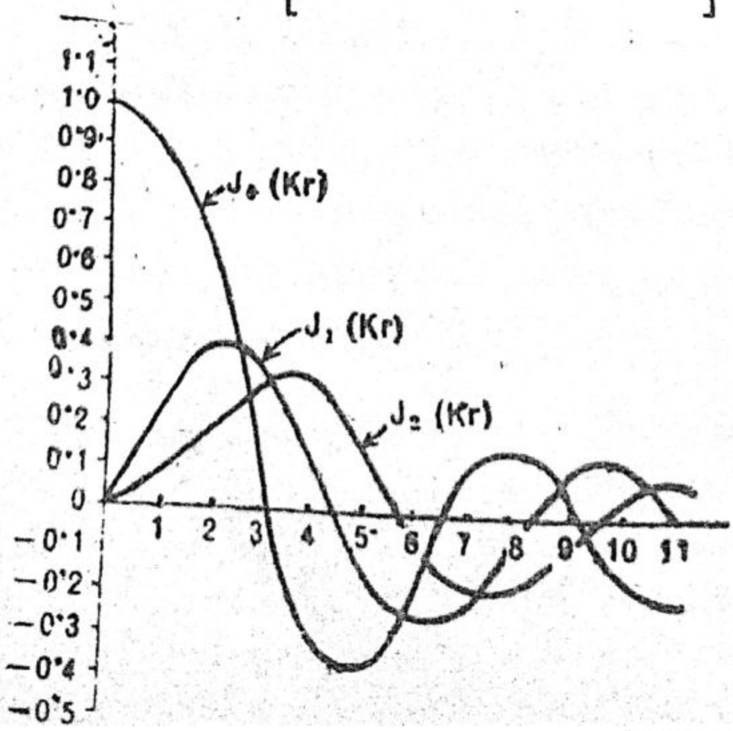
These functions are plotted in the fig. (4·11) below

Similarly $f(\theta)$ may also be expanded in terms of the Legendre Polynomials as follows

$$f(\theta) = \frac{i}{2k}\sum_{l=0}^{\infty} f_l(2l+1)P_l(\cos\theta) \quad \text{...(4·63)}$$

∴ Substituting from equations (4·58), (4·59) and (4·63) in equation (4·57), we obtain

$$\psi = \psi_{inc} + \psi_{sc} \cong \sum_{l=0}^{\infty}\left[i^l(2l+1)j_l(kr) + f_l\frac{e^{ikr}}{r}\right]P_l(\cos\theta) \quad \text{...(4·64)}$$



$$= \frac{\hbar k}{M} = v$$

$$\therefore \quad \sigma(\theta)\, d\Omega = \frac{N_{sc}\, d\Omega}{v\, d\Omega} = \frac{(\hbar\, k/M) |f(\theta)|^2\, d\Omega}{v\, d\Omega} \qquad ...(4\cdot71)$$

or

$$\sigma(\theta) = |f(\theta)|^2 \qquad ...(4\cdot72)$$

With the aid of equation (4·69) the differential scattering cross-section for an S-wave ($l = 0$) is given by

$$\sigma(\theta) = [|f(\theta)|^2]_{l=0} = \frac{e^{2i\delta_0} \cdot \sin^2 \delta_0}{k^2}$$

$$= \frac{\sin^2 \delta_0}{k^2} \qquad \text{Since } e^{2i\delta_0} = 1 \qquad ...(4\cdot73)$$

and the total scattering cross-section for an S-wave is

$$\sigma_{sc,0} = \int \sigma(\theta)\, d\Omega = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{\sin^2 \delta_0}{k^2} \sin\theta\, d\theta\, d\phi$$

$$= 2\pi \int_{\theta=0}^{\pi} \frac{\sin^2 \delta_0}{k^2} \sin\theta\, d\theta$$

$$= 4\pi \frac{\sin^2 \delta_0}{k^2} = 4\pi \lambdabar^2 \sin \delta_0 \qquad ...(4\cdot74)$$

where '$\lambdabar$' is to be read as "Lambda Cross" and $\lambdabar = 1/k$.

Equations (4·73) and (4·74) giving the differential scattering cross-section and total scattering crosssection respectively for $l = 0$ neutrons, show that these are closely related to the phase shift experienced by the outgoing part due to the presence of the scattering potential and are independent of θ. This shows that scattering is isotropic (spherically symmetric) in the CM system, a fact that has been verified experimently at low energies.

When we consider higher energy neutrons, higher orbital angular momentum waves ($l = 1, 2$, etc., *i.e.*, P, D etc. waves) also take part in scattering and so they also need be considered. It can be shown that the total scattering cross-section in general is then given by

$$\sigma = \frac{4\pi}{k^2} \Sigma_{l=0}^{\infty} (2l + 1) \sin^2 \delta_l \qquad ...(4\cdot75)$$

where $2\,\delta_l$ is the phase shift introduced by the scattering centre in the l^{th} orbital angular momentum wave.