3

Magnetostatics

§ 3·1. Current Density J :

In electromagnetic theory, the concept of current density is more useful than the current used in circuit theory. *Current density is defined as the ratio of the current to the surface area whose plane is normal to the direction of charge motion.* It is denoted by **J** and is a vector having direction of charge motion.

Consider a surface ds whose normal **n** is parallel to **v**, the velocity of charge then according to above definition

$$\mathbf{J}=\frac{dI}{ds}\mathbf{n} \quad \ldots(1)$$

or
$$dI\mathbf{n}=\mathbf{J}ds$$

or
$$dI=\mathbf{J}.\mathbf{n}ds$$

$$I=\int_S \mathbf{J}.d\mathbf{s} \quad \ldots(2)$$

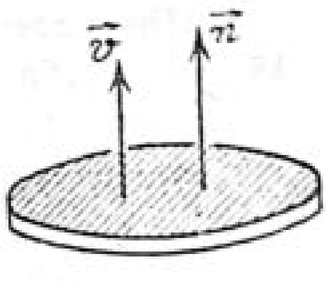
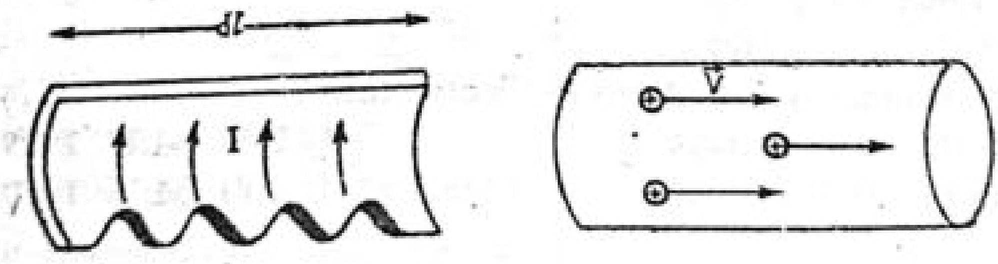


Fig. 3·1

Regarding current density it is worthy to note that

(i) If a current flows in a thin surface layer as shown in fig. 3·2 (a), the so called surface current density is more useful. It is defined by the current per unit length *i.e.*

$$I=\int \mathbf{J}_s.^{*}d\mathbf{l} \quad \ldots(3)$$



(*a*) (*b*)

Fig. 3·2

(ii) It is some times useful to express current density in terms of the velocity of the elementary charges or charge density. If there are n charged particle per unit volume each having a charge e, then

* See example 4 and problem 16.

the charge enclosed in the cylinder of length l and area of cross-section S as shown in fig. 3·2 (b) will

$$q = ne\,(Sl)$$

So the current through cross-sectional area S

$$I = \frac{dq}{dt} = \frac{d}{dt}(ne\,Sl) = neS\frac{dl}{dt}$$

i.e. $$I = ne\,Sv$$

This in turn implies that

$$J = \frac{I}{S} = nev$$

i.e. vectorially $\mathbf{J} = ne\mathbf{v} = \rho\mathbf{v}$ (as $\rho = ne$) ...(4)

(3) When a current I flows in a uniform cross-section S, normally to it :—

$$I = \int \mathbf{J}.d\mathbf{s} = \int J ds \cos 0 = J \int ds = JS$$

or $$Id\mathbf{l} = JS\,d\mathbf{l}$$

or $$Id\mathbf{l} = J\mathbf{n}S\,dl$$

or $$Id\mathbf{l} = \mathbf{J}d\tau \quad (\text{as } J\mathbf{n} = \mathbf{J} \text{ and } Sdl = d\tau) \qquad ...(5)$$

Equation (5) converts the current element $Id\mathbf{l}$ to current density element $\mathbf{J}\,d\tau$.

(4) Ohm's law is $V = IR$.

But as $I = JS$ and $R = l/\sigma S$

So $V = (JS) \times (l/\sigma S)$

i.e. $J = \sigma(V/l)$

But we know that by definition $(V/l) = E$

$$\therefore \quad J = \sigma E$$

or vectorially $\mathbf{J} = \sigma\mathbf{E}$. ...(6)

This is the form of Ohm's law in terms of current density and electric intensity.

(5) As $\mathbf{J} = \sigma\mathbf{E}$

$$\text{curl } \mathbf{J} = \text{curl } (\sigma\mathbf{E})$$

i.e. $\text{curl } \mathbf{J} = \sigma \text{ curl } \mathbf{E}$ [as σ is constant]

i.e. $\text{curl } \mathbf{J} = \sigma \text{ curl } (-\text{grad } V)$ [as $\mathbf{E} = -\text{grad } V$]

i.e. $\text{curl } \mathbf{J} = 0$ [as curl grad $\phi = 0$] ...(7)

i.e. current density is *irrotational*.

(6) By definition

$$I = \int \mathbf{J}.d\mathbf{s}$$

If we consider a closed surface, conservation of charge requires that the net steady current passing through the surface be zero. Thus

$$\oint_S \mathbf{J}.\, d\mathbf{s}=0$$

or $$\int_\tau \nabla.\,\mathbf{J}\, d\tau=0 \qquad \left(\text{as} \oint_S \mathbf{J}.d\mathbf{s}=\int_\tau \nabla.\mathbf{J}\, d\tau\right)$$

Since the law of conservation of charge holds good for any arbitrary vol τ

$$\nabla.\,\mathbf{J}=0 \quad \text{or} \quad \text{div}\,\mathbf{J}=0 \qquad ...(8)$$

Equation (8) is frequently called the eqn. of continuity for steady currents and according to it for steady currents, current density is *solenoidal.*

§ 3·2 (A) Ampere's Law of Force*

Ampere performed a series of experiments to find the force between current carrying conductors. He found that in case of two elements $I_1 d\mathbf{l}_1$ and $I_2 d\mathbf{l}_2$ seperated by a distance r the force between the elements—

(1) varies directly as the product of magnitudes of current

(2) varies inversly as the square of distance between the two current elements.

(3) depends upon the nature of the medium.

(4) depends upon the lengths and orientations of the two current elements.

(5) is attractive if the currents flow in the same direction and repulsive if they flow in opposite direction.

In general case of pair of currents as shown in fig. 3·3 analytically the force which current I_2 exerts on I_1 when both are in free space is given by

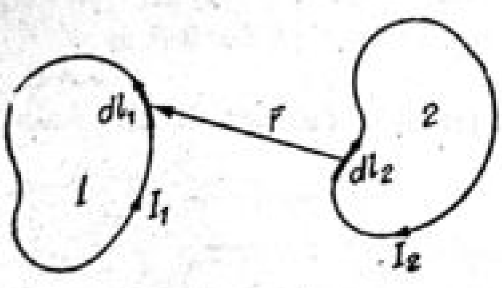


Fig. 3·3

$$\mathbf{F}_{12}=\frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{d\mathbf{l}_1\times(d\mathbf{l}_2\times\mathbf{r})}{r^3} \qquad ...(A)$$

where the line integrals are taken around the two loops and the constant μ_0 is called the *permeability* of free space and is arbitrary taken to be $4\pi\times10^{-7}$ N/amp^2 (or H/m). Eqn. (A) is the mathematical statement of Ampere's observations about forces between current carrying loops and is called *Ampere's law of force.*

* This law is different from Ampere's circuital law which is commonly refered as Ampere's law in lituture and is dealt with in § 3·3.

(B) Biot-Savart law (Definition of magnetic Induction B).

Ampere's force law

$$\mathbf{F}_{12}=\frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{d\mathbf{l}_1 \times (d\mathbf{l}_2 \times \mathbf{r})}{r^3} \quad ...(1)$$

is not of much practical value because the force cannot be expressed as the interaction of current I_1 with the field of current I_2. However eqn. (1) can be written in a more practical forms as

$$F_{12}=I_1 \oint d\mathbf{l}_1 \times \left[\frac{\mu_0}{4\pi} I_2 \oint_2 \frac{d\mathbf{l}_2 \times \mathbf{r}}{r^3}\right]$$

i.e.

$$\mathbf{F}_{12}=\oint I_1 d\mathbf{l}_1 \times \mathbf{B}_2 \quad ...(2)$$

where

$$\mathbf{B}_2=\frac{\mu_0}{4\pi} \oint_2 \frac{I_2 d\mathbf{l}_2 \times \mathbf{r}}{r^3} \quad ...(3)$$

can be taken to be the field of circuit 2 at the position of the element $I_1 d\mathbf{l}_1$ of circuit-1.

The vector **B** is called the *magnetic induction vector*, *magnetic flux density* or *magnetic field*. Its unit is *web/m²** which is sometimes also called *Tesla*.

From equation (3) it is evident that the magnetic induction **B** at position **r** due to a current carrying circuit of element $Id\mathbf{l}$ will be

$$\mathbf{B}=\frac{\mu_0}{4\pi}\int \frac{Id\mathbf{l} \times \mathbf{r}}{r^3} \quad ...(A)$$

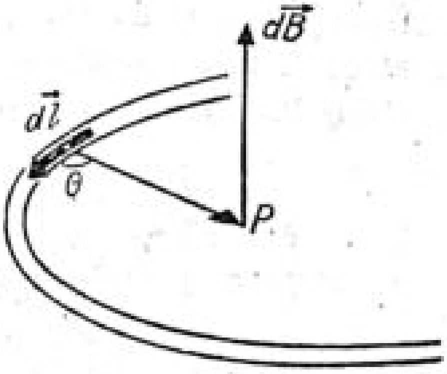


Fig. 3·4

The above equation for **B** is called *Biot-Savart law* or *Laplaces formula*. Regarding Biot-Savart law it is worthy to note that—

(1) It is based on experimental observations of Ampere and is an inverse square law so may be viewed as the magnetic analogue of Coulomb's law and is used to calculate **B** at a point in case of current carrying conductors.

(2) If the current I is distributed in space with a current density **J** then as

$$I\,d\mathbf{l}=\mathbf{J}d\tau \quad \text{[from eqn. 5 of § 3·1]}$$

$$\mathbf{B}=\frac{\mu_0}{4\pi}\int \frac{\mathbf{J} \times \mathbf{r}}{r^3}\, d\tau \quad ...(B)$$

(3) If there is a single charge moving with velocity **v** then as

* Weber = volt-sec. and 1 weber/m² = 10^4 Gauss.

$$I\,d\mathbf{l}=\frac{dq}{dt}\,d\mathbf{l}=dq\frac{d\mathbf{l}}{dt}=\mathbf{v}dq$$

so
$$\mathbf{B}=\frac{\mu_0}{4\pi}\int dq\,\frac{(\mathbf{v}\times\mathbf{r})}{r^3}$$

i.e.
$$\mathbf{B}=\frac{\mu_0}{4\pi}\,q\,\frac{(\mathbf{v}\times\mathbf{r})}{r^3} \qquad ...(1)$$

or
$$\mathbf{B}=\mu_0\epsilon_0\mathbf{v}\times\frac{q\mathbf{r}}{4\pi\epsilon_0 r^3}$$

But as $\mu_0\epsilon_0=\frac{1}{c^2}$ and $\mathbf{E}=\frac{q\mathbf{r}}{4\pi\epsilon_0 r^3}$

so
$$\mathbf{B}=\frac{\mathbf{v}\times\mathbf{E}}{c^2} \qquad ...(2)$$

This is the single and compact relation between the electric and magnetic fields of a uniformly moving charge. And from it, it is clear that as $v << c$, $\mathbf{B} << \mathbf{E}$ *i.e.* the electric field is much stronger than magnetic field, in case of a moving charged particle.

(C) Applications :

(c_1) Long straight wire : P is a point at a fixed distance d from a long wire OQ carrying a current I, where $\mathbf{B}$ is to be evaluated. Let us consider a small element $d\mathbf{l}$ at O as shown in fig. 3·5. Using Biot-Savart law for this element have

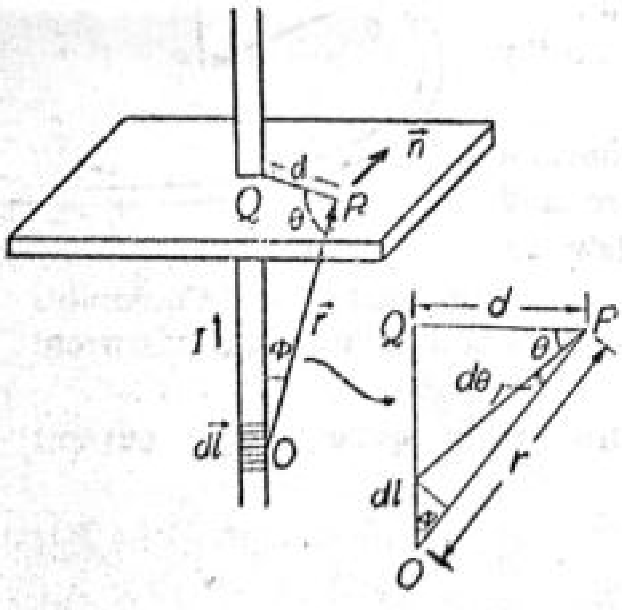


Fig. 3·5

$$d\mathbf{B}=\frac{\mu_0}{4\pi}\frac{I\,d\mathbf{l}\times\mathbf{r}}{r^3}$$

$$d\mathbf{B}=\frac{\mu_0 I}{4\pi}\frac{dl\sin\phi}{r^2}\,\mathbf{n}$$

where $\mathbf{n}$ is a unit vector $\perp$ to the plane containing $I\ d\mathbf{l}$ and $\mathbf{r}$ and is directed into the plane of page.

Now from fig. 3·5

$$dl\sin\phi=r\,d\theta$$

and
$$r=d\sec\theta$$

so
$$\mathbf{B}=\frac{\mu_0 I}{4\pi}\int_{-\pi/2}^{+\pi/2}\frac{\cos\theta}{d}\,d\theta\,\mathbf{n}$$

i.e.
$$\mathbf{B}=\frac{\mu_0}{4\pi}\frac{I}{d}\Big[\sin\theta\Big]_{-\pi/2}^{\pi/2}\mathbf{n}$$

i.e.
$$\mathbf{B}=\frac{\mu_0}{4\pi}\left(\frac{2I}{d}\right)\mathbf{n} \qquad ...(1)$$

This relation indicates that the value of **B** is same for all points at the same distance from the wire. Thus lines of magnetic induction are circles concentric with wire and lying in a plane perpendicular to it.

(c_2) Circular Coil (or loop) :

Let us consider a coil of radius a and carrying a current I. The magnetic field $d\mathbf{B}$ at a point P on its axis at a distance z from the centre, due to a current element of length $d\mathbf{l}$ will be

$$d\mathbf{B}=\frac{\mu_0}{4\pi}\frac{I\,d\mathbf{l}\times\mathbf{r}}{r^3}$$

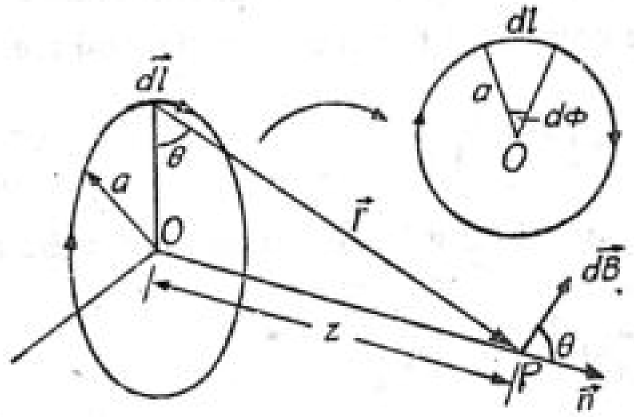


Fig. 3·6

As here the angle between $d\mathbf{l}$ and $\mathbf{r}$ is always 90°

$$dB=\frac{\mu_0}{4\pi}\frac{I\,dl}{r^2} \quad ...(1)$$

Now if the plane of the coil is ⊥ to the plane of paper, the vector $d\mathbf{l}$ for a current element at the loop points perpendicularly out of the page. As for all elements around the loop $\mathbf{r}$ is ⊥ to $d\mathbf{l}$, hence $d\mathbf{B}$ is always ⊥ to the plane containing $\mathbf{r}$ and $d\mathbf{l}$. Thus $d\mathbf{B}$ must lie in the plane of paper as shown in fig. 3·6. It can be resolved into two components one $dB\cos\theta$ along the axis of the coil and the other $dB\sin\theta$ ⊥ to the axis. By symmetry the component of **B** ⊥ to the axis vanishes for complete circuit as the component of $d\mathbf{B}$ due to an element is cancelled by the field due to oppositely situated element. So the resultant **B** is in the direction of the axis and is equal to

$$\mathbf{B}=\int dB\cos\theta\,\mathbf{n} \quad ...(2)$$

where **n** is a unit vector along the axis as shown in fig. 3·6.

So substituting the value of $d\mathbf{B}$ from eqn. (1) in (2)

$$\mathbf{B}=\frac{\mu_0}{4\pi}I\int\frac{dl\cos\theta}{r^2}\,\mathbf{n}$$

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But from fig. 3·6

$$\cos\theta=(a/r) \quad dl=a\,d\phi \quad \text{and } r^2=a^2+z^2$$

So $$\mathbf{B}=\frac{\mu_0}{4\pi}\frac{Ia^2}{(a^2+z^2)^{3/2}}\int_0^{2\pi} d\phi\ \mathbf{n}$$

i.e. $$\mathbf{B}=\frac{\mu_0}{4\pi}\frac{2\pi Ia^2}{(a^2+z^2)^{3/2}}\ \mathbf{n}$$

If there are N turns in the coil $I\to NI$

so $$\mathbf{B}=\frac{\mu_0}{4\pi}\frac{2\pi NIa^2}{(a^2+z^2)^{3/2}}\ \mathbf{n} \quad ...(3)$$

This is the required result and from this it is clear that

(1) If the point is at the centre of the coil *i.e.* $z=0$, eqn (3) reduces to

$$\mathbf{B}=\frac{\mu_0}{4\pi}\left(\frac{2\pi NI}{a}\right)\mathbf{n} \quad ...(4)$$

(2) If the point is at a large distance from the centre of the coil *i.e.* $z >> a$, eqn (3) reduces to

$$\mathbf{B}=\frac{\mu_0}{4\pi}\left(\frac{2\pi NIa^2}{z^3}\right)\mathbf{n}$$

i.e. $$\mathbf{B}=\frac{\mu_0}{4\pi}\left(\frac{2\mathbf{m}}{z^3}\right) \quad ...(5)$$

with $\mathbf{m}=NI(\pi a^2)\,\mathbf{n}=NI\mathbf{S}$ (as $\mathbf{S}=\pi a^2\,\mathbf{n}$) ...(6)

As expression (5) represents the field due to a magnetic dipole of moment $\mathbf{m}$ at distance z on its axis, we come to the conclusion that for a distant point a current carrying current acts as a magnetic dipole of moment $\mathbf{m}=NI\mathbf{S}$.

(c_2) Solenoid.

A solenoid is a long wire wound in a close-packed helix and carrying an electric current.

If $n\ (=N/L)$ is the number of turns per unit length each carrying a current I, uniformly wound round a cylinder of radius a, the number of turns in length dx of solenoid is $n dx$. Thus the magnetic field at the axial point P due to this element dx (treating the element as a coil) is

$$d\mathbf{B}=\frac{\mu_0}{4\pi}\frac{2\pi nIa^2\,dx}{(a^2+x^2)^{3/2}}\ \mathbf{n}$$

where $\mathbf{n}$ is a unit vector along the axis of the solenoid in the sense of advance of a right hand screw. From the geometry of the figure 3·7, we have

$$x = a \cot\theta$$

and $$dx = -a \operatorname{cosec}^2\theta \, d\theta$$

so $$d\mathbf{B} = \frac{\mu_0}{4\pi} [-2\pi n I \sin\theta] \, d\theta \, \mathbf{n}$$

i.e. $$\mathbf{B} = \frac{\mu_0}{4\pi}(2\pi n I) \int_{\theta_1}^{\theta_2} -\sin\theta \; d\theta \; \mathbf{n}$$

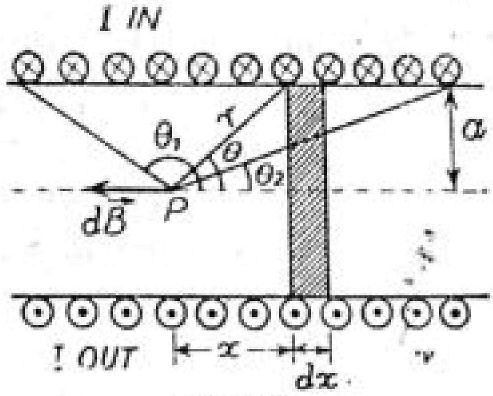

Fig. 3·7

i.e. $$\mathbf{B} = \frac{\mu_0}{4\pi} 2\pi n I [\cos\theta_2 - \cos\theta_1] \; \mathbf{n} \qquad ...(1)$$

Now two cases are possible.

(1) If the solenoid is of infinite length ($L >> a$) and the point P is well within the solenoid

$$\theta_1 \to 180 \qquad \text{and } \theta_2 \to 0$$

so $$\mathbf{B} = \frac{\mu_0}{4\pi} 4\pi n I \, \mathbf{n} = \mu_0 \, n I \, \mathbf{n} \qquad ...(2)$$

(2) If the point P is at one end of solenoid

$$\theta_1 \to 90 \qquad \text{and} \quad \theta_2 \to 0$$

so $$\mathbf{B} = \frac{\mu_0}{4\pi} (2\pi n I) \, \mathbf{n} = \tfrac{1}{2} \mu_0 \, n I \, \mathbf{n} \qquad ...(3)$$

From expressions (2) and (3) it is clear that the field strength at the end of a long solenoid is just one half that at the centre and so half the lines of force passing through the central section of a long solenoid pass out through the sides before reaching the end.

Example 1. *Starting from Biot-Savart law calculate the divergence of magnetic induction vector* **B** (M.U. 1981)

Solution. According to Biot-Savart's law

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l} \times \mathbf{r}}{r^3}$$

so $$\text{div } \mathbf{B} = \nabla \cdot \left[\frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l} \times \mathbf{r}}{r^3} \right]$$

i.e. $$\text{div } \mathbf{B} = \frac{\mu_0 I}{4\pi} \int \nabla \cdot \left(\frac{d\mathbf{l} \times \mathbf{r}}{r^3} \right) \qquad ...(1)$$

But as

$$\text{div } (\mathbf{A} \times \mathbf{C}) = \mathbf{C} \cdot \text{curl } \mathbf{A} - \mathbf{A} \cdot \text{curl } \mathbf{C}$$

i.e. $$\nabla \cdot \left(\frac{d\mathbf{l} \times \mathbf{r}}{r^3} \right) = \frac{\mathbf{r}}{r^3} \cdot \text{curl } d\mathbf{l} - d\mathbf{l} \cdot \text{curl} \frac{\mathbf{r}}{r^3} \qquad ...(2)$$

So equation (1) reduces to

$$\text{div } \mathbf{B}=\frac{\mu_0 I}{4\pi}\int\left(\frac{\mathbf{r}}{r^3}\cdot\text{curl } d\mathbf{l}-d\mathbf{l}\cdot\text{curl}\frac{\mathbf{r}}{r^3}\right)$$

Now as curl $d\mathbf{l}=0$ since $d\mathbf{l}$ is a constant vector and also

$$\text{curl }(\mathbf{r}/r^3)=-\text{curl grad }(1/r) \quad [\text{as grad }(1/r)=-\mathbf{r}/r^3]$$

$$=0 \quad [\text{as curl grad } \phi=0]$$

So $\quad$ $\text{div } \mathbf{B}=0$ ...(3)

i.e. The magnetic field is *solenoidal.*

Note : Alternative proof of $\nabla\cdot\mathbf{B}=0$

As $$\mathbf{B}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}\times\mathbf{r}}{r^3}\,d\tau=\frac{\mu_0}{4\pi}\int-\mathbf{J}\times\nabla\left(\frac{1}{r}\right)d\tau \left[\text{as} \quad \nabla\left(\frac{1}{r}\right)=-\frac{\mathbf{r}}{r^3}\right]$$

$$\nabla\cdot\mathbf{B}=\frac{\mu_0}{4\pi}\int\nabla\cdot\left\{-\mathbf{J}\times\nabla\left(\frac{1}{r}\right)\right\}d\tau$$

But as div $(\mathbf{V}\times\mathbf{B})=\mathbf{B}\cdot\text{curl } \mathbf{V}-\mathbf{V}\cdot\text{curl } \mathbf{B}$

$$\nabla\cdot[\mathbf{J}\times\nabla(1/r)]=\nabla(1/r)\cdot\text{curl } \mathbf{J}-\mathbf{J}\cdot\text{curl }[\nabla(1/r)]$$

i.e. $$\nabla\cdot[\mathbf{J}\times\nabla(1/r)]=-\mathbf{J}\cdot\text{curl }[\nabla(1/r)] \quad [\text{as curl } \mathbf{J}=0]$$

So $$\nabla\cdot\mathbf{B}=\frac{\mu_0}{4\pi}\int\mathbf{J}\cdot\text{curl }\left(\nabla\frac{1}{r}\right)d\tau$$

i.e. $$\nabla\cdot\mathbf{B}=0 \quad [\text{as curl grad } \phi=0]$$

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§ 3·3. Ampere's Circuital Law (Curl B)

According to it for steady currents *the line integral of magnetic induction vector* **B** *around a closed path is equal to* μ_0 *times the total current crossing any surface bounded by the line integral path* ***i.e.***

$$\oint_C \mathbf{B}\cdot d\mathbf{l}=\mu_0\int_S \mathbf{J}\cdot d\mathbf{s}=\mu_0 I.$$

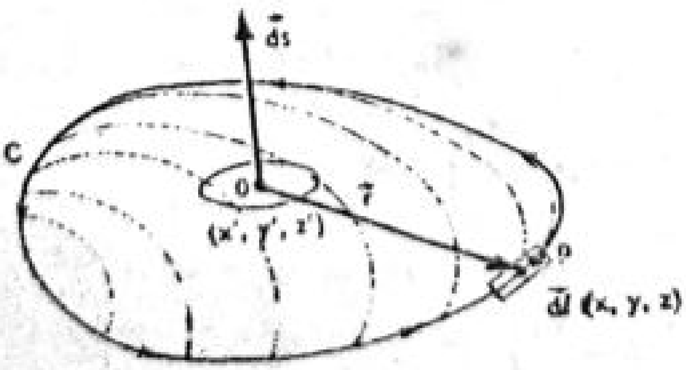

Fig. 3·8

To obtain this law consider a current element $\mathbf{J} \cdot ds$ and a closed path C as shown in fig. 3·8. The element of induction at P due to current at O by Biot-Savart law will be

$$d\mathbf{B}=\frac{\mu_0}{4\pi}\frac{\mathbf{J}'\times\mathbf{r}}{r^3}$$

with $\mathbf{J}'=\mathbf{J}_{(x'\ y'\ z')}$ and $r^2=(x-x')^2+(y-y')^2+(z-z')^2$

So that

$$\mathbf{B}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}'\times\mathbf{r}}{r^3}\,d\tau'$$

i.e. $$\mathbf{B}=\frac{\mu_0}{4\pi}\int \mathbf{J}'\times\nabla\left(-\frac{1}{r}\right)d\tau' \qquad \left[\text{as } \nabla\left(\frac{1}{r}\right)=-\frac{\mathbf{r}}{r^3}\right]$$

But as curl $S\mathbf{V}=S$ curl $\mathbf{V}-\mathbf{V}\times$grad S

$$\text{curl}\,\frac{\mathbf{J}'}{r}=\frac{1}{r}\,\text{curl}\,\mathbf{J}'-\mathbf{J}'\times\nabla\left(\frac{1}{r}\right)$$

[Since $\nabla\times\mathbf{J}'=0$ as $\mathbf{J}'$ is not a function of x, y and z]

$$\therefore\quad \mathbf{B}=\frac{\mu_0}{4\pi}\int\nabla\times\left(\frac{\mathbf{J}'}{r}\right)d\tau'$$

So $$\nabla\times\mathbf{B}=\frac{\mu_0}{4\pi}\int\nabla\times\nabla\times\left(\frac{\mathbf{J}'}{r}\right)d\tau'$$

Now as

$$\nabla\times\nabla\times\mathbf{V}=\text{grad div }\mathbf{V}-\nabla^2\mathbf{V}$$

$$\nabla\times\mathbf{B}=\frac{\mu_0}{4\pi}\int\left[\nabla\left\{\nabla\cdot\left(\frac{\mathbf{J}'}{r}\right)\right\}-\nabla^2\left(\frac{\mathbf{J}'}{r}\right)\right]d\tau'$$

i.e. $$\nabla\times\mathbf{B}=\frac{\mu_0}{4\pi}\int-\nabla^2\left(\frac{\mathbf{J}'}{r}\right)d\tau'+\frac{\mu_0}{4\pi}\nabla\int\nabla\cdot\left(\frac{\mathbf{J}'}{r}\right)d\tau' \quad ...(1)$$

Now as I term in equation (1) *i.e.*

$$-\frac{\mu_0}{4\pi}\int\nabla^2\left(\frac{\mathbf{J}'}{r}\right)d\tau'=-\frac{\mu_0}{4\pi}\int\mathbf{J}'\,\nabla^2\left(\frac{1}{r}\right)d\tau'$$

[as $\mathbf{J}'$ is not a function of x, y and z]

$$=-\frac{\mu_0}{4\pi}\int\mathbf{J}'\,[-4\pi\delta\,(r-r')]\,d\tau'$$

$$\left[\text{as } \nabla^2\left(\frac{1}{r}\right)=-4\pi\delta\,(r-r')\right]$$

$$=\mu_0\,\mathbf{J}' \qquad [\text{as } \textstyle\int\delta\,(r-r')\,d\tau'=1] \quad ...(2)$$

while the II term in equation (1) *i.e.*

$$\frac{\mu_0}{4\pi}\nabla\int\nabla\cdot\left(\frac{\mathbf{J}'}{r}\right)d\tau'$$

$$=\frac{\mu_0}{4\pi}\nabla\int\left[\frac{1}{r}\,\nabla\cdot\mathbf{J}'+\mathbf{J}'\cdot\nabla\left(\frac{1}{r}\right)\right]d\tau'$$

[as $\nabla\cdot(S\mathbf{V})=S\nabla\cdot\mathbf{V}+\mathbf{V}\cdot\nabla S$]

$$=\frac{\mu_0}{4\pi}\nabla\int \mathbf{J}'\bullet\nabla\,(1/r)\,d\tau'$$

[as $\nabla\bullet\mathbf{J}'=0$ as $\mathbf{J}'$ is not a function of x, y and z]

$$=-\frac{\mu_0}{4\pi}\nabla\int \mathbf{J}'\bullet\nabla'\left(\frac{1}{r}\right)d\tau' \quad \left[\text{as } \nabla\left(\frac{1}{r}\right)=-\nabla'\left(\frac{1}{r}\right)\right]$$

$$=-\frac{\mu_0}{4\pi}\nabla\int \nabla'\bullet\left(\frac{\mathbf{J}'}{r}\right)d\tau^*=-\frac{\mu_0}{4\pi}\nabla\oint\left(\frac{\mathbf{J}'}{r}\right).\,d\mathbf{s}$$

$$\left[\text{as } \int\nabla\bullet\mathbf{V}\,d\tau=\oint\mathbf{V}.\,d\mathbf{s}\right]$$

Now as in steady state charge can neither leave nor enter the system, $\mathbf{J}'$ must be either zero or tangential at the boundary. This in tern implies that surface integral must vanish *i.e.*

$$\frac{\mu_0}{4\pi}\nabla\int\nabla\bullet\left(\frac{\mathbf{J}'}{r}\right)d\tau'=0 \qquad ...(3)$$

Substituting eqn. (2) and (3) in (1) we get

$$\nabla\times\mathbf{B}=\mu_0\,\mathbf{J} \quad \text{(dropping the suffix)} \qquad ...(a)$$

This is the *differential form of Ampere's circuital law* and signifies that *magnetic field is rotational*. In order to obtain integral form of this law we integrate eqn. (a) over a surface S bounded by a loop C.

i.e.
$$\int_S(\nabla\times\mathbf{B})\bullet d\mathbf{s}=\mu_0\int_S\mathbf{J}\bullet d\mathbf{s}. \qquad ...(4)$$

But according to Stoke's theorem

$$\int_S(\nabla\times\mathbf{B})\bullet d\mathbf{s}=\oint_C\mathbf{B}\bullet d\mathbf{l} \qquad ...(5)$$

So from (4) and (5)

$$\oint_C\mathbf{B}\bullet d\mathbf{l}=\mu_0\int_S\mathbf{J}\bullet d\mathbf{s}=\mu_0 I \qquad ...(6)$$

---

*As $\nabla\bullet(S\mathbf{V})=S\nabla\bullet\mathbf{V}+\mathbf{V}\bullet\nabla S$

or $\nabla'\left(\frac{\mathbf{J}'}{r}\right)=\frac{1}{r}\nabla'\bullet\mathbf{J}'+\mathbf{J}'\bullet\nabla'\left(\frac{1}{r}\right)$

But as from continuity eqn.

$\nabla'\,\mathbf{J}'+(\partial\rho/\partial t)=0$

for steady state

$\nabla'\bullet\mathbf{J}'=0$ as $\rho=$constt in steady state

so $\nabla'\left(\frac{\mathbf{J}'}{r}\right)=\mathbf{J}'\bullet\nabla'\left(\frac{1}{r}\right)$

This is the required result. This law can be used to compute the magnetic induction for cases in which, by symmetry **B** is constant along some integration path of interest. It is somewhat similar to Gauss's law which is used to compute the electrostatic field intensity **E** when it is constant over a closed surface.

(B) Applications of the law

(*a*) **B** *due to a long straight current carrying conductor.*

Let I be the current through the conductor of radius R. The lines of **B** are concentric circles. Hence the field **B** at point P at a distance r from the axis of wire is given by Ampere's circuital law as

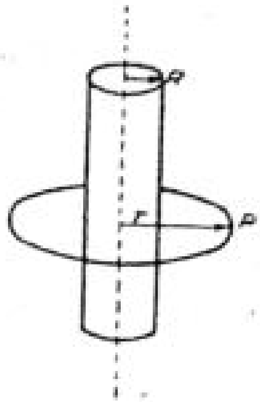

Fig. 3.9.

$$\oint_C \mathbf{B}\cdot d\mathbf{l}=\mu_0\int \mathbf{J}\cdot ds$$

or
$$\mathbf{B}2\pi r=\mu_0\begin{bmatrix}\text{current crosing the}\\ \text{bounded surface}\end{bmatrix} \quad ...(1)$$

(*i*) Now if $r>R$: Obviously current crosing bounded surface is I

so
$$\mathbf{B}\ 2\pi r=\mu_0 I$$

or
$$\mathbf{B}=\frac{\mu_0}{4\pi}\frac{2I}{r} \quad ...(2)$$

This result is same as eqn. (1) in § 3·2 (c_1)

(*ii*) $r<R$. The current crorring the bounded surface will be

$$\mathbf{J}\times\pi r^2=\frac{I}{\pi R^2}\times\pi r^2=I\frac{r^2}{R^2}$$

So from equation (1)

$$\mathbf{B}\times 2\pi r=\mu_0\ I\frac{r^2}{R^2}$$

or
$$\mathbf{B}=\frac{\mu_0}{4\pi}\frac{2I}{R^2}r \quad ...(3)$$

From equations (2) and (3) it is evident that **B** inside a cylindrical conductor varies directly with r while outside the conductor it varies inversely with r.

(*b*) *Field inside a Long Solenoid.*

As the solenoid is long enough and symmetrical so magnetic induction is parallel to the axis of the solenoid by Right hand screw rule.

To determine the value of **B** out 'de the solenoid, consider the path l as shown in Fig. 3·10. From Ampere's law for this path

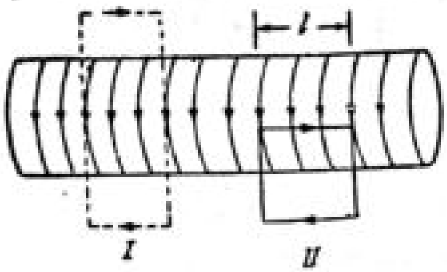

Fig. 3·10.

$$\oint_I \mathbf{B}\cdot d\mathbf{l} = \mu_0 \int (0) = 0$$

But as $d\mathbf{l} \neq 0$ $\quad \mathbf{B} = 0$ ...(1)

i.e. the field outside the solenoid is zero.

To determine the value of **B** inside the solenoid we consider path II for which Ampere's law results in

$$\oint_{II} \mathbf{B}\cdot d\mathbf{l} = \mu_0 NI$$

where N is the number of turns in length L

i.e. $\quad BL = \mu_0 NI$

or $$B = \mu_0 \frac{N}{L} I = \mu_0 nI = \frac{\mu_0}{4\pi}[4\pi nI] \quad ...(2)$$

where n is the number of turns per meter along the solenoid. This result is same as eqn. (2) of § 3·2 (c_3).

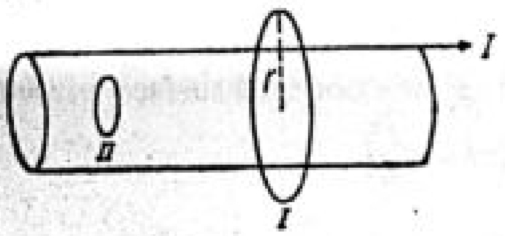

Fig. 3·11

Note : If we have a metal pipe, instead of sole oid for path I

$$\oint_I \mathbf{B}.d\mathbf{l} = \mu_0 I$$

i.e. $B2\pi r = \mu_0 I$ *i.e.* $B = (\mu_0/4\pi)(2I/r)$ while for path II

$$\oint_{II} \mathbf{B}.d\mathbf{l} = 0 \text{ i.e. } B = 0$$

i.e. in a metal pipe, field outside the pipe is $(\mu_0/4\pi)(2I/r)$ while inside is zero. However in case of a solenoid field outside is zero and inside is $(\mu_0/4\pi)[4\pi nI]$

(*c*) **B** *at a point on the axis of a Toroid.*

Toroid is a solenoid bent round in the form of a closed ring. If N is the total number of turns in the toroid and I is the current in each turn, Ampere's circuital law for the circular path shown in fig. 3·12 yields

$$B\, 2\pi r = \mu_0 NI$$

or $B=\mu_0 \left(\frac{NI}{2\pi r}\right)$ ...(1)

However $(NI/2\pi r)=n$, the number of turns per unit length. So

$B=\mu_0 \, nI$ ...(2)

This is the desired result and is same as for a long solenoid *i.e.* eqn. 2 in case (b).

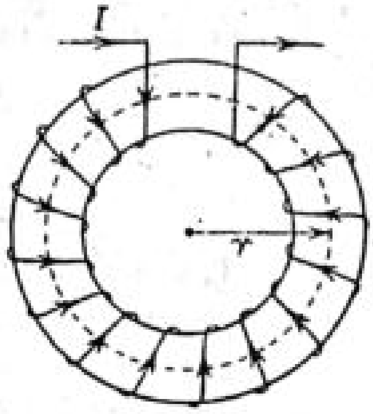

Fig. 3·12

§ 3·4. Force on current carrying conductors and charges :

From Amperes force equation (2) of § 3·2 it is obvious that the force experienced by a current element $I\,d\mathbf{l}$ situated in a magnetic field **B** is given by

$$d\mathbf{F}=I\,d\mathbf{l}\times\mathbf{B}$$

This expression holds even if the maguetic field **B** is produced due to some other current or due to a number of permanent magnets and is used to compute force on a current carrying conductor placed in a magnetic field or a point charge in motion in a magnetic field as follows :

(a) Force between two parallel wires :

Let us now examine the force between two infinite long parallel wires carrying currents I_1 and I_2 separated by a distance d as shown in Fig. 3·13.

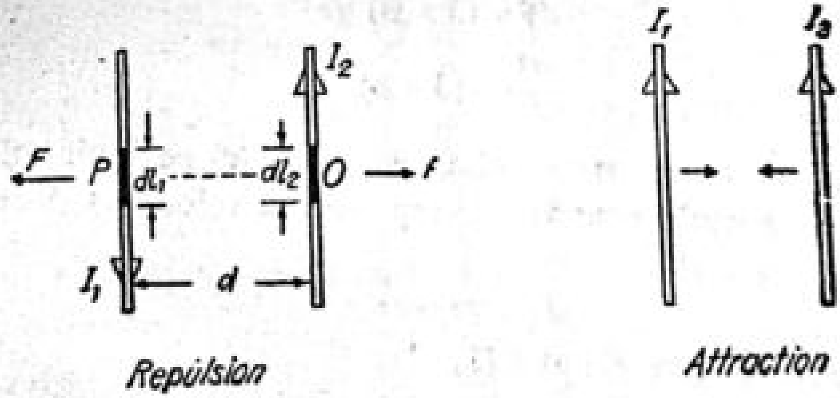

Fig. 3 13.

The current I_2 produces a magnetic induction

$$B=\frac{\mu_0}{4\pi}\frac{2I_2}{d} \text{ at } P \text{ out of page} \quad ...(1)$$

So $\quad d\mathbf{F}=I_1\,d\mathbf{l}_1\times\mathbf{B}=I_1\,dl_1\,B\sin 90$ along OP

i.e. $\quad d\mathbf{F}=I_1\,dl_1 B \quad$ along OP ...(2)

Substituting the value of B from eqn. (1) in (2) we get

$$d\mathbf{F}=I_1\, dl_1 \frac{\mu_0}{4\pi}\frac{2I_2}{d} \text{ along } OP$$

or

$$\frac{dF}{dl_1}=\frac{\mu_0}{4\pi}\left[\frac{2I_1I_2}{d}\right] \quad \ldots(a)$$

This is the required result and from this it is clear that :—

(i) As action and reaction are equal and opposite. The force between two current carrying conductors is repulsive if the current in them is flowing in the opposite directions otherwise attractive.

(ii) If $d=1m$ and $I_1=I_2=1$ *amp*,

$$\frac{d\mathbf{F}}{dl_1}=\frac{\mu_0}{4\pi}\times 2=2\times 10^{-7} \text{ Newton/meter}$$

i.e. One ampere is that current which when flowing through two parallel infinite long straight conductors placed in free space at a distance of 1m apart, produces between them a force of 2×10^{-7} *N/m.*

(b) Force on a point charge moving in a magnetic field.

(Lorentz Force)

The force on a current element $I\, d\mathbf{l}$ in a magnetic field $\mathbf{B}$ is given by

$$d\mathbf{F}=I\, d\mathbf{l}\times\mathbf{B}$$

Now if $\mathbf{J}$ is the current density and ds is the area of cross-section of the conductor normal to current flow

$$I\, d\mathbf{l}=\mathbf{J}\, d\tau$$

so

$$d\mathbf{F}=(\mathbf{J}\times\mathbf{B})\, d\tau$$

i.e.

$$\frac{d\mathbf{F}}{d\tau}=(\mathbf{J}\times\mathbf{B}) \quad \ldots(1)$$

But if there are n charged particles per unit volume, each having a charge q and moving with a velocity $\mathbf{v}$, from eqn. (4) of § 3·1

$$\mathbf{J}=nq\mathbf{v}=\rho\mathbf{v} \quad \ldots(2)$$

So from eqns. (1) and (2)

$$\frac{d\mathbf{F}}{d\tau}=nq\,(\mathbf{v}\times\mathbf{B})=\rho\,[\mathbf{v}\times\mathbf{B}] \quad \ldots(a)$$

Since $d\tau$ is the volume of the charge element, equation (a) express the force per unit volume of the charge carriers. However as in unit volume there are n charges the force experienced by an individual charge will be

$$\mathbf{F}=\frac{1}{n}\frac{d\mathbf{F}}{d\tau}=q\,(\mathbf{v}\times\mathbf{B}) \qquad ...(b)$$

This is required result and is known as *Lorentz-force formula.* However if the charged particle is moving in both electric and magnetic fields, in addition to the above magnetic force it will also experience an electric force $q\mathbf{E}$. In such cases the total force acting on the charged particle will be

$$\mathbf{F}=q\mathbf{E}+q\,(\mathbf{v}\times\mathbf{B})=q\,[\mathbf{E}+(\mathbf{v}\times\mathbf{B})] \qquad ...(c)$$

Example 2. *Find the element of force $d\mathbf{F}_{12}$ caused by the current element $I_2\ d\mathbf{l}_2$ on $I_1\ d\mathbf{l}_1$ as also the force $d\mathbf{F}_{21}$ on the current element $I_2\ d\mathbf{l}_2$ produced by $I_1\ d\mathbf{l}_1$. Show that they are unequal. How do you reconcile your results with Newtons third law ?*

(M.U. 1983)

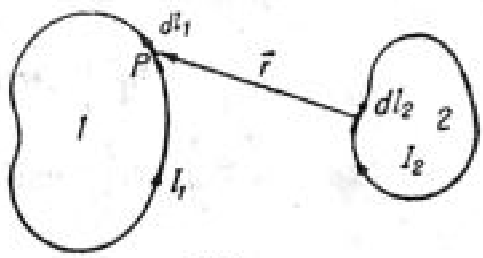

Fig. 3.14

Solution. According to Ampere's force law the force on current element $I_1\ d\mathbf{l}_1$ of circuit 1 due to circuit 2 will be

$$d\mathbf{F}_{12}=I_1\,d\mathbf{l}_1\times\mathbf{B}_2 \qquad ...(1)$$

But according to Boit-Savart law

$$\mathbf{B}_2=\frac{\mu_0}{4\pi}\oint_2\frac{I_2\,d\mathbf{l}_2\times\mathbf{r}}{r^3} \qquad ...(2)$$

So from eqns. (1) and (2)

$$d\mathbf{F}_{12}=\frac{\mu_0}{4\pi}\,I_1I_2\,d\mathbf{l}_1\times\oint_2\frac{(d\mathbf{l}_2\times\mathbf{r})}{r^3}$$

i.e.

$$d\mathbf{F}_{12}=\frac{\mu_0}{4\pi}\,I_1I_2\oint_2\frac{(d\mathbf{l}_1\cdot\mathbf{r})\,d\mathbf{l}_2-(d\mathbf{l}_1\cdot d\mathbf{l}_2)\,\mathbf{r}}{r^3} \qquad ...(a)$$

[as $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\cdot\mathbf{C})\,\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\,\mathbf{C}$]

Similarly the force on current element $I_2d\mathbf{l}_2$ of circuit 2 due to circuit 1 will be

$$d\mathbf{F}_{21}=I_2d\mathbf{l}_2\times\mathbf{B}_1=I_2d\mathbf{l}_2\times\oint_1\frac{\mu_0}{4\pi}\frac{I_1d\mathbf{l}_1\times(-\mathbf{r})}{r^3}$$

i.e.

$$d\mathbf{F}_{21}=\frac{\mu_0}{4\pi}\,I_1I_2\oint_1\frac{d\mathbf{l}_2\times(\mathbf{r}\times d\mathbf{l}_1)}{r^3} \quad [\text{as } (\mathbf{A}\times\mathbf{B})=-(\mathbf{B}\times\mathbf{A})]$$

i.e.

$$d\mathbf{F}_{21}=\frac{\mu_0}{4\pi}\,I_1I_2\oint_1\frac{(d\mathbf{l}_2\cdot d\mathbf{l}_1)\,\mathbf{r}-(d\mathbf{l}_2\cdot\mathbf{r})\,d\mathbf{l}_1}{r^3} \qquad ...(b)$$

Comparing eqns. (a) and (b) we find that

$$d\mathbf{F}_{12} \neq -d\mathbf{F}_{21}$$

This is quite distressing, since from Newtons third law we expect

$$d\mathbf{F}_{12} = -d\mathbf{F}_{21}$$

However if the force due to one complete circuit is evaluated on the other we find that

$$\mathbf{F}_{12} = \oint_1 d\mathbf{F}_{12}$$

i.e.

$$\mathbf{F}_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{(d\mathbf{l}_1 \cdot \mathbf{r})\, d\mathbf{l}_2 - (d\mathbf{l}_1 \cdot d\mathbf{l}_2)\, \mathbf{r}}{r^3}$$

Now as

$$\oint_1 \oint_2 \frac{(d\mathbf{l}_2 \cdot \mathbf{r})\, d\mathbf{l}_2}{r^3} = \oint_2 d\mathbf{l}_2 \oint_1 \frac{(d\mathbf{l}_1 \cdot \mathbf{r})}{r^3}$$

$$= \oint_2 d\mathbf{l}_2 \oint_1 \left[-\nabla\left(\frac{1}{r}\right) \right] \cdot d\mathbf{l}_1 \quad \left[\text{as } \nabla\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3} \right]$$

$$= -\oint_2 d\mathbf{l}_2 \int_{S_1} \text{curl grad} \left(\frac{1}{r}\right) \cdot d\mathbf{s}_1$$

$$\left[\text{as } \oint \mathbf{A} \cdot d\mathbf{l} = \int_S \text{curl } \mathbf{A} \cdot d\mathbf{s} \right]$$

$$= 0 \quad [\text{as curl grad of a scalar} = 0]$$

$$\therefore \quad \mathbf{F}_{21} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{(d\mathbf{l}_1 \cdot d\mathbf{l}_2)\, \mathbf{r}}{r^3} \qquad \text{...(c)}$$

Similarly

$$\mathbf{F}_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{(d\mathbf{l}_2 \cdot d\mathbf{l}_1)\, \mathbf{r} - (d\mathbf{l}_2 \cdot \mathbf{r})\, d\mathbf{l}_1}{r^3}$$

i.e.

$$\mathbf{F}_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{(d\mathbf{l}_2 \cdot d\mathbf{l}_1)\, \mathbf{r}}{r^3} \qquad \text{...(d)}$$

as

$$\oint_1 \oint_2 \frac{(d\mathbf{l}_2 \cdot \mathbf{r})\, d\mathbf{l}_1}{r^3} = -\oint_1 d\mathbf{l}_1 \oint_2 d\mathbf{l}_2 \cdot \text{grad}\left(\frac{1}{r}\right)$$

$$= -\oint_1 d\mathbf{l}_1 \int_{S_2} \text{curl grad} \left(\frac{1}{r}\right) \cdot d\mathbf{s}_2 = 0$$

So from eqns. (c) and (d) it is apparent that

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

§ 3·5. Magnetic Scalar Potential.

Consider a closed loop of current giving rise to an induction $\mathbf{B}$ at P. Let Ω be the solid angle subtanded at P by the loop.

Suppose we displace the point P by the amount $d\mathbf{h}$. Let $d\Omega$ be the change in solid angle subtended by the loop at P resulting from the displacement of P. But we can also get the same change in solid angle $d\Omega$ by keeping P fixed and giving every point of the loop the same but opposite displacement $-d\mathbf{h}$. Then from fig. 3·15 we see that $d\Omega$ equals the sum of solid angles subtanded by each parallelogram formed by $d\mathbf{l}$ and $-d\mathbf{h}$ and therefore equals the sum of projection of these areas along $\mathbf{r}$ with each divided by r^2

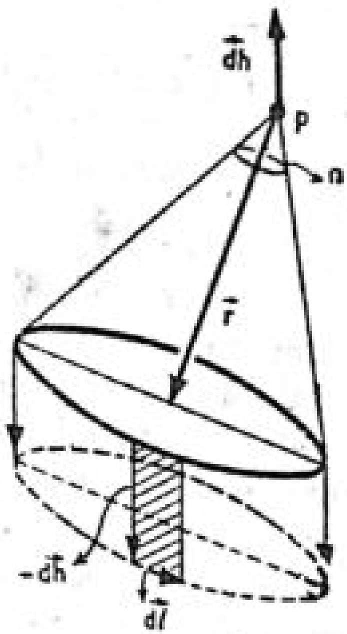

Fig. 3·15.

i.e. $$d\Omega=\Sigma\frac{(-d\mathbf{h}\times d\mathbf{l})\cdot\mathbf{r}}{r^3}$$

or $$d\Omega=-d\mathbf{h}\cdot\int\frac{d\mathbf{l}\times\mathbf{r}}{r^3}$$

Now as for any scalar ϕ

$$\text{grad }\phi=\mathbf{i}\,\frac{\partial\phi}{\partial x}+\mathbf{j}\,\frac{\partial\phi}{\partial y}+\mathbf{k}\,\frac{\partial\phi}{\partial z}$$

and $$d\mathbf{h}=\mathbf{i}dx+\mathbf{j}dy+\mathbf{k}dz$$

$$\therefore\quad(\text{grad }\phi)\cdot d\mathbf{h}=\frac{\partial\phi}{\partial x}\,dx+\frac{\partial\phi}{\partial y}\,dy+\frac{\partial\phi}{\partial z}\,dz=d\phi$$

i.e. $$d\phi=(\text{grad }\phi)\cdot d\mathbf{h}$$

so $$d\Omega=\text{grad }\Omega\cdot d\mathbf{h} \qquad \ldots(2)$$

Comparing equations (1) and (2) we find

$$\text{grad }\Omega=-\int\frac{d\mathbf{l}\times\mathbf{r}}{r^3}. \qquad \ldots(3)$$

Now as the magnetic induction vector $\mathbf{B}$ is given by

$$\mathbf{B}=\frac{\mu_0}{4\pi}\int\frac{I\,(d\mathbf{l}\times\mathbf{r})}{r^3}.$$

So in the light of equation (3) it can be written as

$$\mathbf{B}=\frac{\mu_0 I}{4\pi}\,(-\text{grad }\Omega)$$

i.e. $$\mathbf{B}=-\text{grad}\left(\frac{\mu_0 I}{4\pi}\,\Omega\right)$$

or $$\mathbf{B}=-\text{grad }\Phi_m. \qquad \ldots(4)$$

Thus we see that magnetic induction vector $\mathbf{B}$ can be represented as the negative gradient of a scalar function Φ_m given by

$$\Phi_m = \frac{\mu_0 I}{4\pi}\,\Omega \qquad \text{...(A)}$$

The scalar function Φ_m is called magnetic scalar potential. Regarding scalar potential it is worthy to note that :

(i) It satisfies Laplace equation *i.e.* $\nabla^2 \Phi_m = 0$, This is because as for a magnetic field

$$\nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \mathbf{B} = -\nabla \Phi_m$$

so $\quad \nabla \cdot (-\nabla \Phi_m) = 0 \quad$ *i.e.* $\quad \nabla^2 \Phi_m = 0$

(ii) The concept of magnetic scalar potential leads to a concept of an equivalent magnetic shell, since a sheet of magnetised material whose periphery coincides with the circuit gives the same magnetic potential provided it is every where magnetised perpendicular to its plane with a magnetic moment per unit area equal to current I.

(iii) The concept of magnetic scalar potential makes certain magnetic problems identical in mathematical form with electrostatic one.

(iv) The concept of magnetic scalar potential is severely limited in its applictions as it can be used to derive magnetic fields only in the absence of continuous current distributions. Further this concept connot be used if line integrals encircling any currents are considered, or if the fields within current-carrying media are desired.

Applications :

(a) Φ_m and B for a Magnetic dipole : (Current whirl)

Suppose we have a current I around a small area **S**. P is a point at which Φ_m and **B** are to be computed.

We know that

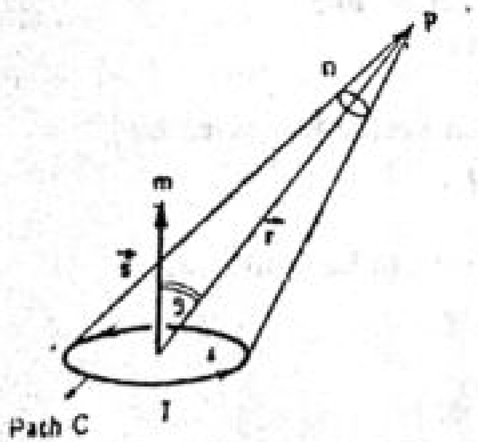

Fig. 3·16

$$\Phi_m = \frac{\mu_0 I}{4\pi}\,\Omega$$

But here $\Omega = \dfrac{\mathbf{S}\cdot\mathbf{r}}{r^3}$

So $\quad \Phi_m = \dfrac{\mu_0 I}{4\pi}\dfrac{\mathbf{S}\cdot\mathbf{r}}{r^3}$

i.e. $\quad \Phi_m = \dfrac{\mu_0 I}{4\pi}\dfrac{\mathbf{S}\cdot\mathbf{r}}{r^3} \qquad \text{...(1)}$

Comparing eqn. (1) with the potential of an electric dipole

$$V = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}$$

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we find that it is appropriate to speak of this small current whirl as *a magnetic dipole* of dipole moment

$$\mathbf{m}=I\mathbf{S}. \qquad \ldots(2)$$

So in the light of eqn. (2), (1) reduces to

$$\Phi_m=\frac{\mu_0}{4\pi}\frac{\mathbf{m}\cdot\mathbf{r}}{r^3}. \qquad \ldots(a)$$

Now

$$\mathbf{B}=-\operatorname{grad}\Phi_m=-\nabla\left(\frac{\mu_0}{4\pi}\frac{\mathbf{m}\cdot\mathbf{r}}{r^3}\right)$$

i.e.

$$\mathbf{B}=-\frac{\mu_0}{4\pi}\left[\frac{1}{r^3}\nabla(\mathbf{m}\cdot\mathbf{r})+(\mathbf{m}\cdot\mathbf{r})\nabla\left(\frac{1}{r^3}\right)\right]$$

But

$$\nabla(\mathbf{m}\cdot\mathbf{r})=\nabla(m_x x+m_y y+m_z z)=\mathbf{i}m_x+\mathbf{j}m_y+\mathbf{k}m_z=\mathbf{m}$$

and $\nabla\left(\frac{1}{r^3}\right)=\Sigma\mathbf{i}\frac{\partial}{\partial x}[(x-x')^2+(y-y')^2+(z-z')^2]^{-3/2}$

$$=-3\,\Sigma\frac{\mathbf{i}(x-x')}{r^5}=\frac{-3\mathbf{r}}{r^5}$$

so

$$\mathbf{B}=-\frac{\mu_0}{4\pi}\left[\frac{\mathbf{m}}{r^3}-\frac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r}}{r^5}\right]$$

i.e.

$$\mathbf{B}=\frac{\mu_0}{4\pi}\left[\frac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r}}{r^5}-\frac{\mathbf{m}}{r^3}\right] \qquad \ldots(b)$$

Note : (1) If the point is on the axis of the loop $\theta=0$

so $\mathbf{B}=\frac{\mu_0}{4\pi}\left[\frac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r}}{r^5}-\frac{\mathbf{m}}{r^3}\right]=\frac{\mu_0}{4\pi}\frac{2\mathbf{m}}{r^3}$

This result is same as eqn. 5 of § 3·2 c_2

(2) In case of a moving charged particle as

$$I=q\times(v/2\pi r)$$

so $\mathbf{m}=I\mathbf{S}=(qv/2\pi r)\pi r^2\mathbf{n}=(qvr/2)\,\mathbf{n} \qquad \ldots(1)$

However the angular momentum

$$\mathbf{L}=\mathbf{r}\times\mathbf{p}=mvr\mathbf{n} \qquad \ldots(2)$$

So from eqns. (1) and (2)

$$\mathbf{m}=\frac{q}{2m}\mathbf{L}$$

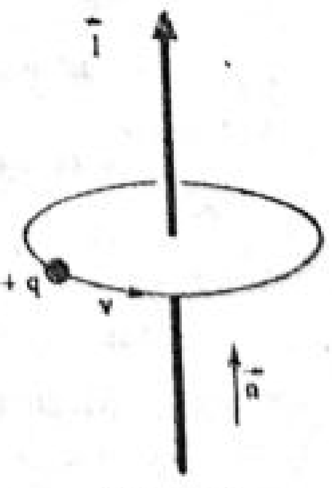

Fig. 3·17

This result is of great importance when we attempt to calculate the magnetic moment of elementary particles. Further the quantity $(q/2m)$ for an electron is called gyromagnetic ratio and plays important role in atomic physics.

(b) Φ_m and B for a Circular coil

As $$d\Omega = \frac{ds}{r^2}$$

for this case

$$\Omega = \int \frac{ds}{r^2} = \int_0^\theta \frac{(2\pi r \sin\alpha)\, r d\alpha}{r^2}$$

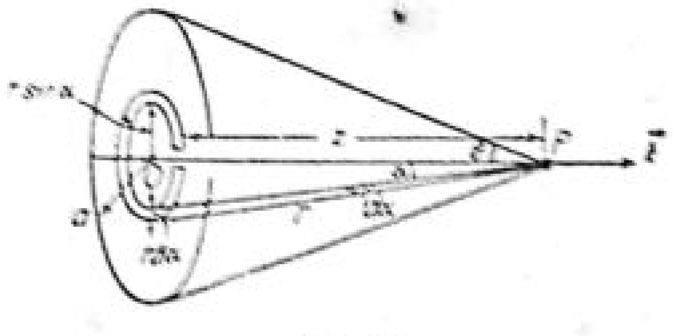

Fig. 3·18.

i.e. $$\Omega = 2\pi \Big[-\cos\theta \Big]_0^\theta = 2\pi\,(1-\cos\theta)$$

i.e. $$\Omega = 2\pi\left[1 - \frac{z}{(a^2+z^2)^{1/2}}\right] \qquad \left(\text{as } \cos\theta = \frac{z}{(a^2+z^2)^{1/2}}\right)$$

so $$\Phi_m = \frac{\mu_0 I}{4\pi}\Omega = \frac{\mu_0}{4\pi}\, 2\pi I\left[1 - \frac{z}{(a^2+z^2)^{1/2}}\right] \qquad \ldots(a)$$

and $$\mathbf{B} = -\nabla\Phi_m = -\mathbf{k}\frac{\partial}{\partial z}\frac{\mu_0}{4\pi}\, 2\pi I\left[1 - \frac{z}{(a^2+z^2)^{1/2}}\right]$$

(as Φ_m is a function of z only)

i.e. $$\mathbf{B} = -\frac{\mu_0}{4\pi}\, 2\pi I\left[-\frac{1}{(a^2+z^2)^{1/2}} - \left(-\tfrac{1}{2}\right)\frac{z\cdot 2z}{(a^2+z^2)^{3/2}}\right]\mathbf{k}$$

i.e. $$\mathbf{B} = \frac{\mu_0}{4\pi}\left[\frac{2\pi I a^2}{(a^2+z^2)^{3/2}}\right]\mathbf{k}$$

If there are N turns in the coil, $I \to NI$

so $$\mathbf{B} = \frac{\mu_0}{4\pi}\left[\frac{2\pi N I a^2}{(a^2+z^2)^{3/2}}\right]\mathbf{k}$$

This is the required result and is same as eqn. (3) of § 3·2 (c_2).

§ 3·6. Magnetic Vector Potential

We know that magnetic induction vector **B** according to Biot-Savart's law is given by

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int I\,\frac{d\mathbf{l}\times\mathbf{r}}{r^3}$$

i.e. $$\mathbf{B}=\frac{\mu_0 I}{4\pi}\int d\mathbf{l}\times\left\{-\text{grad}\left(\frac{1}{r}\right)\right\} \qquad \left[\text{as } \nabla\left(\frac{1}{r}\right)=\frac{-\mathbf{r}}{r^3}\right]$$

But if vector **A** is not a function of position

$$\text{Curl } S\mathbf{A}=S \text{ curl } \mathbf{A}-\mathbf{A}\times\text{grad } S$$

i.e. $$-\mathbf{A}\times\text{grad } S=\text{curl } S\mathbf{A}-S \text{ curl } \mathbf{A}$$

So $$\mathbf{B}=\frac{\mu_0 I}{4\pi}\int\left[\text{curl}\frac{d\mathbf{l}}{r}-\frac{1}{r}\text{curl }(d\mathbf{l})\right]$$

The second term in the integral is zero because $d\mathbf{l}$ does not depend on the co-ordinates of the field point ,while the first term on changing the order of differentiation and integration yields

$$\mathbf{B}=\text{curl}\int\frac{\mu_0 I\, d\mathbf{l}}{4\pi r}$$

or $$\mathbf{B}=\text{curl } \mathbf{A}. \qquad ...(A)$$

Thus we see that magnetic induction vector **B** can be represented as the curl of a vector function **A** given by

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{I\, d\mathbf{l}}{r}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}\, d\tau}{r} \text{ (as } I\, d\mathbf{l}=\mathbf{J}\, d\tau). \qquad ...(B)$$

The vector function **A** is called magnetic vector potential. Regarding magnetic vector potential **A** it must be noted that—

(*i*) *It satifies Poisson's equation* :

By definition, as

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}'}{r}\, d\tau'$$

$$\nabla^2\mathbf{A}=\frac{\mu_0}{4\pi}\int\nabla^2\left(\frac{\mathbf{J}'}{r}\right)d\tau'=\frac{\mu_0}{4\pi}\int\mathbf{J}'\,\nabla^2\left(\frac{1}{r}\right)d\tau'$$

(as $\mathbf{J}'$ is not a function of x, y, and z)

or $$\nabla^2\mathbf{A}=-\mu_0\,\mathbf{J}' \qquad \left[\text{as } \int\nabla^2(1/r)\, d\tau'=-4\pi\right]$$

(*ii*) *The line integral of magnetic vector* **A** *round a closed path gives the magnetic flux linked with the area enclosed by the closed path i.e.*

$$\Phi_B=\oint\mathbf{A}\cdot d\mathbf{l}$$

By definition

$$\Phi_B=\int_S\mathbf{B}\cdot d\mathbf{s}=\int_S\text{curl } \mathbf{A}\cdot d\mathbf{s} \qquad [\text{as } \mathbf{B}=\text{curl } \mathbf{A}]$$

i.e. $$\Phi_B=\oint\mathbf{A}\cdot d\mathbf{l} \qquad \left[\text{as} \int\text{curl } \mathbf{A}\cdot d\mathbf{s}=\oint\mathbf{A}\cdot d\mathbf{l}\right]$$

(*iii*) *The divergence of magnetic vector potential* **A** *is zero or a scalar constant*.

By definition of **A**

$$\mathbf{B}=\text{curl } \mathbf{A}$$

so $$\nabla\times\mathbf{B}=\nabla\times\nabla\times\mathbf{A}=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$$

But as

$$\nabla\times\mathbf{B}=\mu_0\,\mathbf{J} \quad \text{and} \quad \nabla^2\mathbf{A}=-\mu_0\,\mathbf{J}$$

$$\nabla(\nabla\cdot\mathbf{A})=0 \quad \textit{i.e.} \quad \text{grad div } \mathbf{A}=0$$

This means that

i.e. div **A** is zero or forms a uniform scalar field with constant value.*

(iv) There are essentially no cases where **A** can be computed in simple form. The principle use of this concept is in approximations made in problems related to electromagnetic relations.

Applications.

(a) A and B for a magnetic dipole.

We know that

$$\mathbf{A}=\frac{\mu_0 I}{4\pi}\int\frac{d\mathbf{l}}{r}.$$

But as according to Stroke's theorem†

$$\int_S(-\text{grad } S)\times d\mathbf{s}=\oint_C S d\mathbf{l}$$

So $$\mathbf{A}=-\frac{\mu_0 I}{4\pi}\int_S \text{grad}\left(\frac{1}{r}\right)\times d\mathbf{s}$$

---

* For details see example—3.

†Strokes theorem is

$$\int_S \text{curl } \mathbf{A}\cdot d\mathbf{s}=\oint_C \mathbf{A}\cdot d\mathbf{l}$$

If $\mathbf{A}=S\mathbf{B}$ where **B** is a constant vectors it reduces to

$$\int_S \text{curl } (S\mathbf{B})\cdot d\mathbf{s}=\oint_C S\mathbf{B}\cdot d\mathbf{l} \qquad ...(1)$$

But as curl $(S\mathbf{B})=S$ curl $\mathbf{B}-\mathbf{B}\times\text{grad } S$

i.e. curl $(S\mathbf{B})=-\mathbf{B}\times\text{grad } S$ (as **B** is a constant vector, curl $\mathbf{B}=0$)

equation (1) reduces to

$$-\int_S(\mathbf{B}\times\text{grad } S)\cdot d\mathbf{s}=\oint_C S\mathbf{B}\cdot d\mathbf{l}$$

or $$\int_S(-\text{grad } S)\times d\mathbf{s}=\oint_C S d\mathbf{l}$$

The gradient is clearly taken here w.r.t. the source coordinates, and so we must change the sign if we want **A** in terms of field coordinates. Moreover if the circuit is small compared with the distance r the gradient of $1/r$ will not change appreciably over the surface S and in the limit may be taken outside the integral.

Hence $$\mathbf{A}=\frac{\mu_0}{4\pi}\,\text{grad}\left(\frac{1}{r}\right)\times\int_S I d\mathbf{s} \qquad ...(1)$$

But as

$$\text{grad}\left(\frac{1}{r}\right)=-\frac{\mathbf{r}}{r^3} \text{ and } \int_S I\, d\mathbf{s}=\mathbf{m}$$

$$\mathbf{A}=\frac{\mu_0}{4\pi}\left(-\frac{\mathbf{r}}{r^3}\right)\times\mathbf{m}$$

or $$\mathbf{A}=\frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3} \text{ (as} -\mathbf{r}\times\mathbf{m}=\mathbf{m}\times\mathbf{r}) \qquad ...(a)$$

Further $\mathbf{B}=\nabla\times\mathbf{A}=\nabla\times\left[\frac{\mu_0}{4\pi}\left(\frac{\mathbf{m}\times\mathbf{r}}{r^3}\right)\right]$.

But as

curl $(\mathbf{A}\times\mathbf{B})=\mathbf{A}$ div $\mathbf{B}-(\mathbf{A}\cdot\text{grad})\,\mathbf{B}$, for constant **A**

So $$\mathbf{B}=\frac{\mu_0}{4\pi}\left[\mathbf{m}\text{ div}\frac{\mathbf{r}}{r^3}-(\mathbf{m}\cdot\text{grad})\frac{\mathbf{r}}{r^3}\right].$$

Now as $$m_x\frac{\partial}{\partial x}\left(\frac{\mathbf{r}}{r^3}\right)=\frac{m_x\mathbf{i}}{r^3}-3m_x x\frac{\mathbf{r}}{r^5}$$

i.e. $$(\mathbf{m}\cdot\text{grad})\frac{\mathbf{r}}{r^3}=\frac{\mathbf{m}}{r^3}-\frac{3(\mathbf{m}\cdot\mathbf{r})\,\mathbf{r}}{r^5}.$$

And $$\mathbf{m}\text{ div}\frac{\mathbf{r}}{r^3}=\mathbf{m}\left[\frac{3}{r^3}-\mathbf{r}\cdot\frac{3\mathbf{r}}{r^5}\right]=0$$

So $$\mathbf{B}=\frac{\mu_0}{4\pi}\left[\frac{3(\mathbf{m}\cdot\mathbf{r})\,\mathbf{r}}{r^5}-\frac{\mathbf{m}}{r^3}\right] \qquad ...(b)$$

(b) A and B for a long current carrying wire :

Consider a long wire of length L. If P is a point at a distance y from the wire, the magnetic vector potential at P due to the current element $I\,d\mathbf{l}$ will be given by

$$d\mathbf{A}=\frac{\mu_0 I}{4\pi}\frac{d\mathbf{l}}{r}$$

or $$d\mathbf{A}=\frac{\mu_0 I}{4\pi}\left(\frac{dx}{r}\right) \qquad ...(1)$$

where **i** is a unit vector parallel to the wire in the direction of current flow.

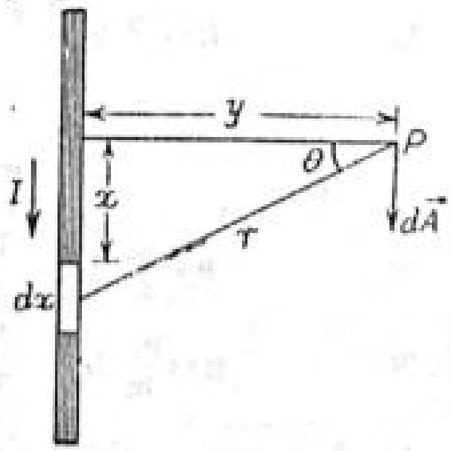

Fig 3·19.

The magnetic vector potential at P due to the whole wire will be obtained by integrating equation (1) between the limits $(-L/2)$ to $(+L/2)$

i.e. $$\mathbf{A}=\frac{\mu_0 I}{4\pi}\int_{-L/2}^{+L/2}\left(\frac{dx}{r}\right)\mathbf{i}$$

$$\therefore \quad \mathbf{A}=\frac{\mu_0 I}{4\pi}\,\mathbf{i}\int_{-L/2}^{+L/2}\frac{dx}{\sqrt{(x^2+y^2)}}^{*} \qquad (\text{as } r^2=x^2+y^2)$$

or $$\mathbf{A}=\frac{\mu_0 I}{4\pi}\,\mathbf{i}\left[\log\{x+\sqrt{(x^2+y^2)}\}\right]_{-L/2}^{+L/2}$$

or $$\mathbf{A}=\frac{\mu_0 I}{4\pi}\,\mathbf{i}\log\left[\frac{\frac{L}{2}+\sqrt{\left(\frac{L^2}{4}+y^2\right)}}{-\frac{L}{2}+\sqrt{\left(\frac{L^2}{4}+y^2\right)}}\right] \qquad \ldots(2)$$

If the wire is sufficiently long so that $L^2 >> y^2$ equation (2) reduces to

$$\mathbf{A}=\frac{\mu_0 I}{4\pi}\,\mathbf{i}\log\left[\frac{1+\left(1+\frac{4y^2}{L^2}\right)^{1/2}}{-1+\left(1+\frac{4y^2}{L^2}\right)^{1/2}}\right]$$

or $$\mathbf{A}=\frac{\mu_0 I}{4\pi}\,\mathbf{i}\log\left[\frac{2+(2y^2/L^2)}{(2y^2/L^2)}\right]$$

or $$\mathbf{A}=\frac{\mu_0 I}{4\pi}\log\left[\frac{L^2}{y^2}+1\right]\mathbf{i}$$

or $$\mathbf{A}=\frac{\mu_0 I}{4\pi}\,2\log\frac{L}{y}\,\mathbf{i}\ (\text{as } L >> y) \qquad \ldots(a)$$

Now as $\mathbf{B}=\text{curl}\,\mathbf{A}=\text{curl}\left[\frac{\mu_0}{4\pi}2I\log\left(\frac{L}{x}\right)\right]$

i.e. $$\mathbf{B}=\frac{\mu_0}{4\pi}2I\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \log\frac{L}{y} & 0 & 0\end{vmatrix}$$

or $$\mathbf{B}=\frac{\mu_0}{4\pi}2I\left[-\frac{\partial}{\partial y}\left\{\log\left(\frac{L}{y}\right)\right\}\right]\mathbf{k}$$

or $$\mathbf{B}=\frac{\mu_0}{4\pi}2I\left[-\frac{\partial}{\partial y}(\log L-\log y)\right]\mathbf{k}$$

$$^{*}\int\frac{dx}{\sqrt{(x^2+a^2)}}=\sinh^{-1}\left(\frac{x}{a}\right)=\log\,[x+\sqrt{(x^2+a^2)}]$$

or $$\mathbf{B}=\frac{\mu_0}{4\pi}\left(\frac{2I}{y}\right)\mathbf{k} \quad \text{...(b)}$$

Example 3. *Show that through its definition magnetic vector potential* **A** *is related to electrostatic potential V through the relation*

$$\text{div }\mathbf{A}+\epsilon_0\mu_0\frac{\partial V}{\partial t}=0.$$

Solution. By definition $\mathbf{A}=\frac{\mu_0}{3\pi}\int\frac{\mathbf{J}'}{r}d\tau'$

so $$\text{div }\mathbf{A}=\frac{\mu_0}{4\pi}\int\nabla\cdot\left(\frac{\mathbf{J}'}{r}\right)d\tau'$$

$$=\frac{\mu_0}{4\pi}\int\left[\frac{1}{r}\nabla\cdot\mathbf{J}'+\mathbf{J}'\cdot\nabla\left(\frac{1}{r}\right)\right]d\tau'$$

[as $\nabla\cdot(S\mathbf{V})=S\nabla\cdot\mathbf{V}+\mathbf{V}\cdot\nabla S$]

$$=\frac{\mu_0}{4\pi}\int\mathbf{J}'\cdot\nabla\left(\frac{1}{r}\right)d\tau'$$

[as $\mathbf{J}'$ is not a function of x, y and z]

$$=-\frac{\mu_0}{4\pi}\int\mathbf{J}'\cdot\nabla'\left(\frac{1}{r}\right)d\tau' \quad \left[\text{as } \nabla\left(\frac{1}{r}\right)=-\nabla'\left(\frac{1}{r}\right)\right] \quad \text{...(1)}$$

Now as $\nabla'\cdot(S\mathbf{V})=S\nabla'\cdot\mathbf{V}+\mathbf{V}\cdot\nabla'S$

i.e. $$\nabla'\cdot\left(\frac{\mathbf{J}'}{r}\right)=\frac{1}{r}\nabla'\cdot(\mathbf{J}')+\mathbf{J}'\cdot\nabla'\left(\frac{1}{r}\right)$$

i.e. $$\mathbf{J}'\cdot\nabla'\left(\frac{1}{r}\right)=\nabla'\cdot\left(\frac{\mathbf{J}'}{r}\right)-\frac{1}{r}\nabla'\cdot(\mathbf{J}') \quad \text{...(2)}$$

so from equation (1) and (2)

$$\text{div }\mathbf{A}=-\frac{\mu_0}{4\pi}\left[\int\nabla'\cdot\left(\frac{\mathbf{J}'}{r}\right)d\tau'-\int\frac{\nabla'\cdot\mathbf{J}'}{r}d\tau'\right] \quad \text{...(3)}$$

Now converting the volume integral into surface integral by Gauss's theorem we have

$$\int\nabla'\cdot(\mathbf{J}'/r)\,d\tau'=\oint(\mathbf{J}'/r)\cdot d\mathbf{s}'$$

$$\therefore \quad \text{div }\mathbf{A}=-\frac{\mu_0}{4\pi}\left[\oint\left(\frac{\mathbf{J}'}{r}\right)\cdot d\mathbf{s}'-\int\frac{\nabla'\cdot\mathbf{J}'}{r}d\tau'\right]$$

Now as $\mathbf{J}'$ is confined to vol τ', the surface contribution must vanish

so $$\text{div }\mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{\nabla'\cdot\mathbf{J}'}{r}d\tau'$$

or $$\text{div }\mathbf{A}=-\frac{\mu_0}{4\pi}\int\frac{1}{r}\frac{\partial\rho}{\partial t}d\tau'$$

[as by continuity equation $\nabla'\cdot\mathbf{J}'=-(\partial\rho/\partial t)$]

or $\quad \text{div } \mathbf{A} = -\mu_0\epsilon_0 \frac{\partial}{\partial t}\int \frac{\rho}{4\pi\epsilon_0 r}\, d\tau'$

or $\quad \text{div } \mathbf{A} = -\mu_0\epsilon_0 \frac{\partial V}{\partial t}$ $\qquad \left[\text{as } V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho}{r}\, d\tau'\right]$

or $\quad \text{div } \mathbf{A} + \mu_0\epsilon_0 \frac{\partial V}{\partial t} = 0$ ...(4)

Hence proved.

Note : This result is called Lorentz condition and is discussed in § 4·10. Further as $\mu_0\epsilon_0 = (1/c^2)$ above condition can be written as

$$\text{div } \mathbf{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t} \qquad ...(5)$$

In case of steady state V = constt, *i.e.* $(\partial V/\partial t) = 0$

so $\quad \text{div } \mathbf{A} = 0 \qquad ...(6)$

Eqns. (5) and (6) are in accordance with (iii) of § 3·6.

Example 4. *A large number N of closely spaced turns of fine wire are wound in a single layer upon the surface of a wooden sphere of radius a, with the planes of the turns perpendicular to the axis of the sphere and completely covering its surface. If the current in the winding is I, determine the vector potential* **A** *and the field* **B** *at the centre of the sphere.* **(M.U. 1975)**

Solution. We know that $\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}\, d\tau}{r}$

But as the current flows only in surface $\mathbf{J}\, d\tau = \mathbf{J}_s\, ds$ for this case

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}_s\, ds}{r}$$

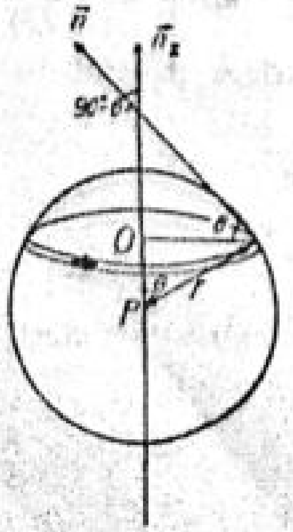

Fig. 3·20.

Now since the magnitude of $\mathbf{J}_s$ is uniform throughout the surface, it is obvious from the symmetry that

$$\mathbf{A} = 0 \qquad ...(a)$$

But $\quad d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{\mathbf{J}_s \times \mathbf{r}}{r^3}\, ds$

i.e. $\quad d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{J}{a^2}\, ds\, \mathbf{n}$ (as $\mathbf{J} \times \mathbf{r} = Ia \sin 90$)

It is obvious once again from symmetry considerations that only that component of **B** will contribute to **B** which is parallel to z-axis

so $\quad \mathbf{B} = \frac{\mu_0 J}{4\pi}\int \frac{\sin\theta}{a^2}\, ds\, \mathbf{n}_z$

i.e. $$\mathbf{B}=\frac{\mu_0 J}{4\pi}2\pi\int_0^{\pi}\frac{\sin\theta}{a^2}(2\pi a\sin\theta)\,a\,d\theta\,\mathbf{n}_z$$

i.e. $$\mathbf{B}=\frac{\mu_0 J}{4\pi}2\pi\int_0^{\pi}\sin^2\theta\,d\theta\,\mathbf{n}_z$$

i.e. $$\mathbf{B}=\frac{\mu_0}{4\pi}2\pi J\int_0^{\pi}\tfrac{1}{2}(1-\cos 2\theta)\,d\theta\,\mathbf{n}_z$$

i.e. $$\mathbf{B}=\frac{\mu_0}{4\pi}2\pi J\mathbf{n}_z\left[\tfrac{1}{2}\left(\theta-\frac{\sin 2\theta}{2}\right)\right]_0^{\pi}$$

i.e. $$\mathbf{B}=\frac{\mu_0}{4\pi}2\pi J\mathbf{n}_z\left(\frac{\pi}{2}\right)=\frac{\mu_0}{4\pi}J\pi^2\mathbf{n}_z$$

But here $J=(NI/\pi a)$

so $$\mathbf{B}=\frac{\mu_0}{4\pi}\frac{NI}{\pi a}\pi^2\mathbf{n}_z=\frac{\mu_0}{4\pi}\left[\frac{NI\pi}{a}\right]\mathbf{n}_z$$

§ 37. Multiple Expansion of a current distribution or Vector potential A

For a given current distribution as shown in fig. 3·21 the vector potential **A** is

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}\,d\tau'}{R}=\frac{\mu_0}{4\pi}\int\frac{I\,d\mathbf{r}'}{R}$$ [as $\mathbf{J}\,d\tau'=I\,d\mathbf{r}'$] ...(1)

But from fig. 3·21

$$\frac{1}{R}=\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\frac{1}{(r^2+r'^2-2rr'\cos\theta)^{1/2}}$$

i.e. $$\frac{1}{R}=\frac{1}{r}\left[1-\frac{2r'}{r}\cos\theta+\frac{r'^2}{r^2}\right]^{-1/2}$$

i.e. $$\frac{1}{R}=\frac{1}{r}\left[1-2z\cos\theta+z^2\right]^{-1/2}$$ with $z=r'/r$.

Now as $(1-2z\cos\theta+z^2)^{-1/2}$ is the generating function of Legender polynomial i.e.

$$(1-2z\cos\theta+z^2)^{-1/2}=P_{(n)}(\cos\theta)\,z^n$$

So $$\frac{1}{R}=\frac{1}{r}\left[P_{(n)}(\cos\theta)\left(\frac{r'}{r}\right)^n\right]$$

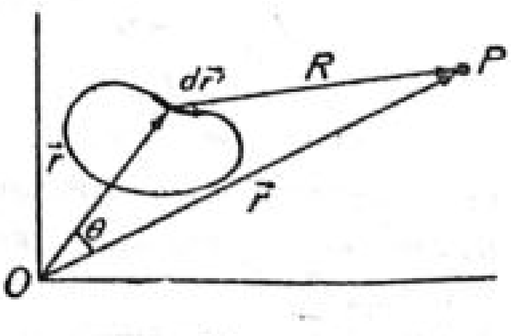

Fig. 3·21.

i.e. $$\frac{1}{R}=\frac{1}{r}\left[1+\cos\theta\left(\frac{r'}{r}\right)^1+\tfrac{1}{2}(3\cos^2\theta-1)\left(\frac{r'}{r}\right)^2+\ldots\right]$$

i.e. $$\frac{1}{R}=\left[\frac{1}{r}+\frac{r'}{r^2}\cos\theta+\frac{r'^2}{r^3}\,\tfrac{1}{2}(3\cos^2\theta-1)+\ldots\right]$$

i.e. $$\frac{1}{R}=\left[\frac{1}{r}+\frac{\mathbf{r}'.\mathbf{r}}{r^3}+\ldots\right] \quad \ldots(2)$$

So subitituting the value of $(1/R)$ from eqn. (2) in (1) we get

$$\mathbf{A}=\frac{\mu_0}{4\pi r}\oint I\,d\mathbf{r}'+\frac{\mu_0}{4\pi r^3}\oint I\,(\mathbf{r}'.\mathbf{r})\,d\mathbf{r}'+\ldots \quad \ldots(3)$$

This is the required result. From this it is clear that

(1) As the vector sum of elements $d\mathbf{r}'$ round a closed loop is zero *i.e.* $\oint I\,d\mathbf{r}'=0$; the first term *i.e. monopole term is zero.* This conclusion agrees with the fact that *free magnetic poles do not exist.*

(2) As for a scalar S

$$\oint S\,d\mathbf{l}=-\int \nabla S\times d\mathbf{s}$$

[See foot note in § 3·6 application (a)]

i.e. $$\oint (\mathbf{r}'\cdot\mathbf{r})\,d\mathbf{r}'=-\int \nabla'(\mathbf{r}'\cdot\mathbf{r})\times d\mathbf{s}$$

so the second term in eqn. (a) reduces to

$$=-\frac{\mu_0}{4\pi r^3}\int I\,\nabla'(\mathbf{r}'\cdot\mathbf{r})\times d\mathbf{s}$$

Now as

$$\nabla'(\mathbf{r}'.\mathbf{r})=\Sigma\mathbf{i}\,\frac{\partial}{\partial x'}(x\,x')=\Sigma\,\mathbf{i}\,x=\mathbf{r}$$

The above term reduces to

$$=-\frac{\mu_0}{4\pi r^3}\int I\,\mathbf{r}\times d\mathbf{s}=-\frac{\mu_0}{4\pi r^3}\,\mathbf{r}\times\int I\,d\mathbf{s}$$

$$=-\frac{\mu_0}{4\pi r^3}\,\mathbf{r}\times\mathbf{m} \qquad \left[\text{as} \int I\,d\mathbf{s}=\mathbf{m}\right]$$

$$=\frac{\mu_0}{4\pi}\frac{(\mathbf{m}\times\mathbf{r})}{r^3}$$

(as $\mathbf{r}\times\mathbf{m}=-\mathbf{m}\times\mathbf{r}$)

i.e. the second term represents potential due to magnetic dipole and as the first i.e. monopole term is zero. It is the leading term in the expansion.

$$Z=\frac{2\pi r-x}{\mu S}+\frac{x}{\mu_0 S'} \qquad \ldots(1)$$

while m.m.f$=NI\approx 2\pi rnI$ ...(2)

so the flux

$$\phi_B=\frac{2\pi rnI}{\left[\frac{2\pi r-x}{\mu S}+\frac{x}{\mu_0 S'}\right]} \qquad \ldots(3)$$

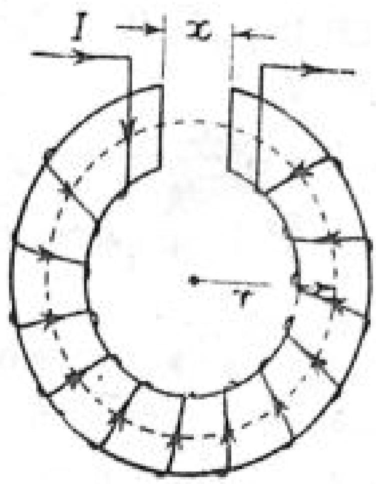

Fig. 3·30

When the thickness x of the air gap is very small compared to the diameter of the pole faces the spreading of the flux in the air gap is negligible so that $S=S'$ and hence the flux density at any point in the magnetic circuit will be

$$B=\frac{2\pi rnI}{\left[\frac{\pi 2r-x}{\mu}+\frac{x}{\mu_0}\right]} \qquad \ldots(4)$$

Further since $\mu \gg \mu_0$ for Iron, the reluctance of the circuit is largely provided by the air gap, unless the air gap is very narrow.

Symmetries in Electrostatics and Magnetostatics.

Electrostatics	Magnetostatics
(1) $\mathbf{E}=\frac{1}{4\pi\epsilon_0}\int\frac{\rho\mathbf{r}}{r^3}\,d\tau$	(1) $\mathbf{B}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}\times\mathbf{r}}{r^3}\,d\tau$
(2) div $\mathbf{E}=\frac{\rho}{\epsilon_0}$	(2) div $\mathbf{B}=0$
(3) curl $\mathbf{E}=0$	(3) curl $\mathbf{A}=\mu_0\,\mathbf{J}$
(4) $V=\frac{1}{4\pi\epsilon_0}\int\frac{\rho d\tau}{r}$	(4) $\mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}\,d\tau}{r}$
(5) $\mathbf{E}=-$grad V	(5) $\mathbf{B}=-$grad Φ_m
(6) $\mathbf{F}=q\mathbf{E}$	(6) $\mathbf{F}=q\,(\mathbf{v}\times B)$
(7) $\rho'=-$div $\mathbf{P}$	(7) $\mathbf{J}_M=$curl $\mathbf{M}$
(8) $\sigma'=\mathbf{P}\cdot\mathbf{n}$	(8) $\mathbf{J}_{SM}=\mathbf{M}\times\mathbf{n}$
(9) $\epsilon_r=\epsilon/\epsilon_0$	(9) $\mu_r=\mu/\mu_0$
(10) $\epsilon_r=1+\chi_e$	(10) $\mu_r=1+\chi_m$
(11) $\mathbf{P}=\epsilon_0\chi_e\mathbf{E}$	(11) $\mathbf{M}=\chi_m\mathbf{H}$
(12) $\mathbf{D}=\epsilon\mathbf{E}$	(12) $\mathbf{B}=\mu\mathbf{H}$

(13) $\mathbf{D}=\epsilon_0\mathbf{E}+\mathbf{P}$	(13) $\mathbf{B}=\mu_0(\mathbf{H}+\mathbf{M})$
(14) $\mathbf{p}=q\mathbf{r}$	(14) $\mathbf{m}=I\mathbf{S}$
(15) $V=\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}$	(15) $V=\frac{\mu_0}{4\pi}\frac{\mathbf{m}\cdot\mathbf{r}}{r^3}$
(16) $\mathbf{E}=\left[\frac{-\mathbf{p}}{r^3}+\frac{3(\mathbf{p}\cdot\mathbf{r})\mathbf{r}}{r^5}\right]$	(16) $\mathbf{B}=\left[\frac{-\mathbf{m}}{r^3}+\frac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r}}{r^5}\right]$
(17) $\vec{\tau}=\mathbf{p}\times\mathbf{E}$	(17) $\vec{\tau}=\mathbf{m}\times\mathbf{B}$
(18) $U_p=-\mathbf{p}\cdot\mathbf{E}$	(18) $U_m=-\mathbf{m}\cdot B$
(19) $U=\frac{1}{2}\int\rho V\,d\tau$	(19) $U=\frac{1}{2}\int\mathbf{J}\cdot\mathbf{A}\,d\tau$
(20) $U=\frac{1}{2}\int\mathbf{E}\cdot\mathbf{D}\,d\tau$	(20) $U=\frac{1}{2}\int\mathbf{H}\cdot B\,d\tau$
(21) $u_e=\frac{1}{2}\epsilon E^2$	(21) $u_m=\frac{1}{2}\mu H^2$

Correspondences in Electrostatics and Magnetostatics

Electrostatics	Magnetostics
(1) Coulomb's Law $\mathbf{E}=\frac{1}{4\pi\epsilon_0}\int\frac{\rho\,\mathbf{r}}{r^3}\,d\tau$	(1) Boit-Savart Law $\mathbf{B}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}\,d\tau}{r^3}$
(2) Gausses Law $\oint\mathbf{E}.d\mathbf{s}=\frac{1}{\epsilon_0}\int_\tau\rho\,d\tau$	(2) Amperes Law $\oint\mathbf{B}\cdot d\mathbf{l}=\mu_0\int\mathbf{J}\cdot d\mathbf{s}$
(3) $\mathbf{D}=\epsilon\mathbf{E}$ $\mathbf{D}=\epsilon_0\mathbf{E}+\mathbf{P}$ $u_e=\frac{1}{2}\epsilon E^2$	(3) $\mathbf{H}=\frac{1}{\mu}\mathbf{B}$ $\mathbf{H}=\frac{1}{\mu_0}\mathbf{B}-\mathbf{M}$ $u_m=\frac{1}{2}\frac{1}{\mu}B^2$

Note. It is worthnoting that **E** corresponds to **B** as both depends on the medium while **D** corresponds to **H** as both do not depend on the medium.