IV - PLANE ELECTROMAGNETIC WAVES & WAVE PROPAGATION

8.11. The Wave Equation

We shall now derive the equations for electromagnetic waves be the use of Maxwell's equations. This is one of the most important applications of Maxwell's equations.

Let us consider a uniform linear medium having permitivity ε, permeability μ and conductivity σ; but not any charge or any current other than that determined by Ohm's law. Then

$$\mathbf{D} = \varepsilon\mathbf{E}\,;\ \mathbf{B} = \mu\mathbf{H}\,;\ \mathbf{J} = \sigma\mathbf{E} \quad \text{and} \quad \rho = 0. \qquad \text{...(1)}$$

So the Maxwell's equations

$$\left.\begin{aligned} \text{div}\,\mathbf{D} &= \rho \\ \text{div}\,\mathbf{B} &= 0 \\ \text{curl}\,\mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \text{and} \qquad \text{curl}\,\mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \end{aligned}\right\} \qquad \text{...(2)}$$

in this case take the form

$$\text{div}\,\mathbf{E} = 0 \qquad \text{...(3)}$$

$$\text{div}\,\mathbf{H} = 0 \qquad \text{...(4)}$$

$$\text{curl}\,\mathbf{E} = -\mu\frac{\partial \mathbf{H}}{\partial t} \qquad \text{...(5)}$$

and

$$\text{curl}\,\mathbf{H} = \sigma E + \varepsilon\frac{\partial \mathbf{E}}{\partial t} \qquad \text{...(6)}$$

Taking curl of equation (5), we get

$$\text{curl curl}\,\mathbf{E} = -\mu\frac{\partial}{\partial t}(\text{curl}\,\mathbf{H})$$

Substituting curl **H** from equation (6), we get

$$\text{curl curl}\,\mathbf{E} = -\mu\frac{\partial}{\partial t}\left(\sigma\mathbf{E} + \varepsilon\frac{\partial \mathbf{E}}{\partial t}\right)$$

i.e.

$$\text{curl curl}\,\mathbf{E} = -\sigma\mu\frac{\partial \mathbf{E}}{\partial t} - \varepsilon\mu\frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \text{...(7)}$$

Similarly, if we take the curl of equation (6) and substitute **E** from equation (5), we obtain

$$\text{curl curl}\,\mathbf{H} = -\sigma\mu\frac{\partial \mathbf{H}}{\partial t} - \varepsilon\mu\frac{\partial^2 \mathbf{H}}{\partial t^2} \qquad \text{...(8)}$$

Now using vector identity

$$\text{curl curl } \mathbf{A} = \text{grad div } \mathbf{A} - \nabla^2\mathbf{A}$$

and keeping in view equations (3) and (4) (*i.e.*, div $\mathbf{E} = 0$ and div $\mathbf{H} = 0$) ; equation (7) and (8) take the form

$$\nabla^2\mathbf{E} - \sigma\mu \frac{\partial \mathbf{E}}{\partial t} - \varepsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

and

$$\nabla^2\mathbf{H} - \sigma\mu \frac{\partial \mathbf{H}}{\partial t} - \varepsilon\mu \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0. \quad \text{...(10)}$$

Equations (9) and (10) represent wave equations which govern the electromagnetic field in a homogeneous, linear medium in which the charge density is zero; whether this medium is conducting or non-conducting. However, it is not enough that these equations be satisfied; Maxwell's equations must also be satisfied. It is clear that equations (9) and (10) are consequence of Maxwell's equations; but the converse is not true. Now the problem is to solve wave equations (9) and (10) in such a manner the Maxwell's equations are also satisfied. One method that works very well for *monochromatic wave (i.e.* waves characterised by a single frequency) is to obtain a solution for **E.** Then curl **E** will give time derivative of **B** $\left(\text{since curl } E = -\frac{\partial \mathbf{B}}{\partial t}\right)$ so that **B** can be computed.

It is more convenient to use the method of complex variable analysis for the solution of wave equations. The time dependence of the field (for certainty we take vector **E**) is taken to be $e^{-i\omega t}$, so that

$$\mathbf{E}(\mathbf{r}, t) = E_S(\mathbf{r})\, e^{-i\omega t} \quad \text{...(11)}$$

It may be noted that the *physical electric field is obtained by taking the real part of* (11) : furthermore $\mathbf{E}_S(\mathbf{r})$ is in general complex so that the actual electric field is proportional to cos $(\omega t + \phi)$, where ϕ is phase of $\mathbf{E}_S(\mathbf{r})$. Using equation (11), equation (9) (droping common factor $e^{-i\omega t}$) gives that

$$\nabla^2\mathbf{E}_S + \omega^2\varepsilon\mu\, \mathbf{E}_S + i\omega\sigma\mu\, \mathbf{E}_S = 0 \quad \text{...(12)}$$

Here the spatial electric field $\mathbf{E}_S$ depends on the space co-ordinates *i.e.*

$$\mathbf{E}_S = \mathbf{E}_S(\mathbf{r}).$$

For plane electromagnetic waves it is convenient to put

$$\mathbf{E}_S = \mathbf{E}_0\, e^{i\mathbf{k}\cdot\mathbf{r}}$$

where **k** is the propagation wave vector defined as

$$\mathbf{k} = \frac{2\pi}{\lambda}\mathbf{n} = \frac{\omega}{v}\mathbf{n}, \ \mathbf{n} \text{ being unit vector along } \mathbf{k}$$

and **r** is position vector from origin, **v** is the phase velocity of the wave.

With this in mind, equation (11) may be written as

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \quad \text{...(13)}$$

Here $\mathbf{E}_0$ is complex amplitude and is constant in space and time. *It is important to note that when field vector is in form* (13), *i.e.* operation of grad, div and curl on field vector is equivalent to

$$\left.\begin{array}{l}\text{grad} \to i\mathbf{k}\,;\ \text{div} = \nabla\cdot - i\,\mathbf{k}\cdot\,;\ \text{curl} = \nabla\times \to i\mathbf{k}\times \\ \text{Also } \frac{\partial}{\partial t} \to -i\omega.\end{array}\right\} \quad \text{...(14)}$$

Now we shall conside various cases of interest to determine field vectors **E** and **H** in electromagnetic field.

8.12. Plane Electromagnetic Waves in Free Space.

Maxwell's equaitons are

$$\left.\begin{aligned} \text{div } \mathbf{D} &= \nabla \cdot \mathbf{D} = \rho \\ \text{div } \mathbf{B} &= \nabla \cdot \mathbf{B} = 0 \\ \text{curl } \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \text{and} \quad \text{curl } \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \end{aligned}\right\} \text{ and } \begin{aligned} \mathbf{B} &= \mu \mathbf{H} \\ \mathbf{D} &= \varepsilon \mathbf{E} \\ \mathbf{J} &= \sigma \mathbf{E} \end{aligned} \qquad \ldots(1)$$

Free space is characterised by

$$\rho = 0, \ \sigma = 0, \ \mu = \mu_0 \text{ and } \varepsilon = \varepsilon_0 \qquad \ldots(2)$$

Therefore Maxwell's equations reduce to

$$\left.\begin{aligned} \text{div } \mathbf{E} &= 0 & \ldots(a) \\ \text{div } \mathbf{H} &= 0 & \ldots(b) \\ \text{curl } \mathbf{E} &= -\mu_0 \frac{\partial \mathbf{H}}{\partial t} & \ldots(c) \\ \text{and} \quad \text{curl } \mathbf{H} &= \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} & \ldots(d) \end{aligned}\right\} \qquad \ldots(3)$$

Taking curl of equation 3(c), we get

$$\text{curl curl } \mathbf{E} = -\mu_0 \frac{\partial}{\partial t} (\text{curl } \mathbf{H})$$

Substituting curl **H** from [3(d)], we get

$$\text{curl curl } \mathbf{E} = \mu_0 \frac{\partial}{\partial t} \left(\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

i.e.

$$\text{curl curl } \mathbf{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \ldots(4)$$

Now

$$\text{curl curl } \mathbf{E} = \text{grad div } \mathbf{E} - \nabla^2 \mathbf{E}$$

i.e.

$$\text{curl curl } \mathbf{E} = -\nabla^2 \mathbf{E} \qquad [\text{since div } \mathbf{E} = 0 \text{ from 3(a)}]$$

Making this substitution equation (4) becomes

$$\nabla^2 \mathbf{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \qquad \ldots(5)$$

Now taking curl of equation [3(d)], we get

$$\text{curl curl } \mathbf{H} = \varepsilon_0 \frac{\partial}{\partial t} (\text{curl } \mathbf{E}).$$

Substituting curl from [3(c)], we get

$$\text{curl curl } \mathbf{H} = \varepsilon_0 \frac{\partial}{\partial t} \left(-\mu_0 \frac{\partial \mathbf{H}}{\partial t} \right) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{H}}{\partial t^2} \qquad \ldots(6)$$

Again using identity curl curl **H** = grad div **H** − ∇^2**H** and noting that div **H** = 0 from [3(b)], we obtain

$$\text{curl curl } \mathbf{H} = -\nabla^2 \mathbf{H}.$$

Making this substitution in equation (6), we get

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$$\nabla^2 \mathbf{H} - \mu_0\varepsilon_0 \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0. \qquad ...(7)$$

Equations (5) and (7) represent *wave equations governing electromagnetic fields* **E** *and* **H** *in free space.* It may be noted that these equations may be obtained by using (2) in equations (9) and (10) of preceding section. Equations (5) and (7) are vector equations of identical form which means that each of the six components of **E** and **H** separately satisfies the same scalar wave equation of the form

$$\nabla^2 u - \mu_0\varepsilon_0 \frac{\partial^2 u}{\partial t^2} = 0 \qquad ...(8)$$

where u is a scalar *and can stand for one of the components of* **E** and **H**. It is obvious that equation (8) resembles with the general wave equation

$$\nabla^2 u = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} \qquad ...(9)$$

where v is the velocity of wave.

Comparing (8) and (9), we see that *the field vectors* **E** *and* **H** *are propagated in free space as waves at a speed equal to*

$$v = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \qquad \text{Since } \mu_0 = 4\pi \times 10^{-7} \text{ weber/Amp–m}$$

$$\varepsilon_0 = 8{\cdot}542 \times 10^{-12} \text{ farad/m}$$

$$= \sqrt{\left(\frac{4\pi}{\mu_0 . 4\pi\varepsilon_0}\right)}$$

So that $\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9$ m/farad.

$$= \sqrt{\left(\frac{4\pi}{4\pi \times 10^{-7}} \times 9 \times 10^9\right)}$$

$$= 3 \times 10^8 \text{ m/sec} = c, \textit{ the speed of light.}$$

Therefore it is reasonable to write c the speed of light in place of $\frac{1}{\sqrt{\mu_0\varepsilon_0}}$; so equations (5) and (7) take the form

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \qquad ...(10)$$

$$\nabla^2 \mathbf{H} - \frac{1}{c} \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \qquad ...(11)$$

and

$$\nabla^2 u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0. \qquad ...(12)$$

Now let us find the solution of above equations for plane electromagnetic waves. ***A plane wave is defined as a wave whose amplitude is the same at any point in a plane perpendicular to a specified direction.***

The plane wave solutions of above equations in well known form may be written as

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \qquad ...(13)$$

$$\mathbf{H}(\mathbf{r}, t) = \mathbf{H}_0\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \quad ...(14)$$

$$u(\mathbf{r}, t) = u_0\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \quad ...(15)$$

where $\mathbf{E}_0$, $\mathbf{H}_0$ and u_0 are complex amplitudes which are constant in space and time while $\mathbf{k}$ is a wave propagation vector denoted as

$$\mathbf{k} = k\mathbf{n} = \frac{2\pi}{\lambda}\,\mathbf{n} = \frac{2\pi v}{c}\,\mathbf{n} = \frac{\omega}{c}\,\mathbf{n} \quad ...(16)$$

Here **n** is a unit vector in the direction of wave propagation. Now in order to apply the conditions $\nabla\cdot\mathbf{E} = 0$ and $\nabla\cdot\mathbf{H} = 0$, let us first find $\nabla\cdot\mathbf{E}$ and $\nabla\cdot\mathbf{H}$.

$$\nabla\cdot\mathbf{E} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot \mathbf{E}_0\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot [(\hat{i}E_{0x} + \hat{j}E_{0y} + \hat{k}E_{0z})\, e^{i(k_x x + k_y y + k_z z) - i\omega t}]$$

$$[\text{since } \mathbf{k}\cdot\mathbf{r} = (\hat{i}k_x + \hat{j}k_y + \hat{k}k_z)\cdot(\hat{i}x + \hat{j}y + \hat{k}z)$$

$$= [k_x x + k_y y + k_z z]$$

$$\therefore \quad \nabla\cdot\mathbf{E} = (E_{0x}\, ik_x + E_{0y}\, ik_z\, E_0)\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$$

$$= i(k_x E_{0x} + k_y E_{0y} + k_z E_{0z})\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$$

$$= i(\hat{i}k_x + \hat{j}k_y + \hat{k}k_z)\cdot(\hat{i}E_{0x} + \hat{j}E_{0y} + \hat{k}E_{0z})\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$$

$$= i\mathbf{k}\cdot\mathbf{E}_0\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} = i\mathbf{k}\cdot\mathbf{E}$$

Similarly

$$\nabla\cdot\mathbf{H} = i\mathbf{k}\cdot\mathbf{H}$$

Thus the requirements $\nabla\cdot\mathbf{E} = 0$ and $\nabla\cdot\mathbf{H} = 0$ demand that

$$\mathbf{k}\cdot\mathbf{E} = 0 \text{ and } \mathbf{k}\cdot\mathbf{H} = 0 \quad ...(17)$$

This means that *electromagnetic field vectors* **E** *and* **H** *are both perpendicular to the direction of propagation vector* **k**. *This implies that* ***electromagnetic waves are transverse in character.*** Further restrictions are provided by curl equations (3c) and 3(d) viz.

$$\text{curl}\,\mathbf{E} = -\mu_0\frac{\partial\mathbf{H}}{\partial t} \quad \text{and} \quad \text{curl}\,\mathbf{H} = \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

using equations (13) and (14), above equations yield

$$i\mathbf{k}\times\mathbf{E} = -\mu_0\cdot(-i\omega\mathbf{H}) \quad \text{or} \quad \mathbf{k}\times\mathbf{E} = \mu_0\omega\mathbf{H} \quad ...(18)$$

and

$$i\mathbf{k}\times\mathbf{H} = \varepsilon_0\cdot(-i\omega\mathbf{E}) \quad \text{or} \quad \mathbf{k}\times\mathbf{H} = -\varepsilon_0\omega\mathbf{E}. \quad ...(19)$$

From equation (18) it is obvious that field vector **H** is perpendicular to both **k** and **E** and according to equation (19) **E** perpendicular to both **k** and **H**. This simply means that *field vectors* **E** *and* **H** *are mutually perpendicular and also they are also perpendicular to the direction of propagation of wave.* This all in turn implies that in a plane electromagnetic wave, vectors (**E, H, k**) form a set of orthogonal vectors which form a right handed co-ordinate system in tha order (fig. 8.2).

Further from equation (18).

$$\mathbf{H} = \frac{1}{\mu_0\omega}(\mathbf{k}\times\mathbf{E}) = \frac{k}{\mu_0\omega}(\mathbf{n}\times\mathbf{E}) \quad (\text{since } \mathbf{k} = k\mathbf{n})$$

$$= \frac{1}{\mu_0 c}(\mathbf{n}\times\mathbf{E}) \quad ...(20)$$

This equation in term of moduli

$$H = \frac{1}{\mu_0 c} E \left(\text{since } \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c \right)$$

Now the ratio of magnitude of **E** to the magnitude of **H** is symbolised as Z_0 *i.e.*

$$Z_0 = \left| \frac{\mathbf{E}}{\mathbf{H}} \right| = \left| \frac{E_0}{H_0} \right| = \mu_0 c = \sqrt{\left(\frac{\mu_0}{\varepsilon_0} \right)}$$

$$\left(\text{since } c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \right)$$

$$= \sqrt{\left(\frac{4\pi \times 10^{-7}}{8{\cdot}85 \times 10^{-12}} \right)} = \mathbf{376{\cdot}6\ ohms} \qquad ...(21)$$

where the units of Z_0 are most easily seen from the fact that it measures a ratio of E in volt/m to H in amp-turn/m and therefore must equal volt/amp or Ohms. Because the units of E/H are the same as those of impedance, the value of Z_0 is often referred to as the **wave impedance** of free

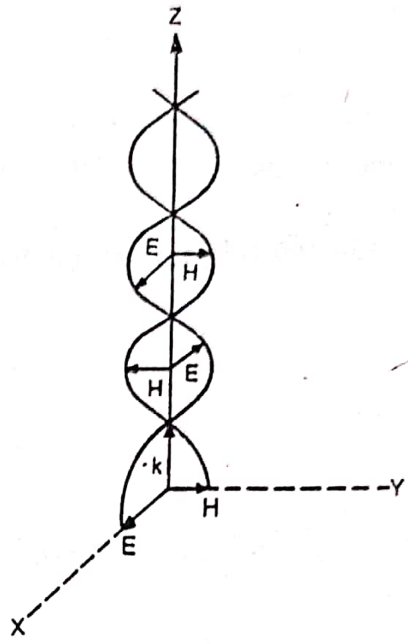

Fig. 8.2

space. Further since the ratio $Z_0 = \left| \frac{\mathbf{E}}{\mathbf{H}} \right|$ is real and positive ; this implies *that field vectors* **E** *and* **H** *are in the same phase i.e.* they have the same relative magnitude at all points at all times (fig. 8.2).

The Poynting vector (i.e. energy flow per unit area per unit time) for a plane electromagnetic wave is given by

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \mathbf{E} \times \frac{\mathbf{n} \times \mathbf{E}}{\mu_0 c} \qquad \text{using (20)}$$

$$= \frac{1}{\mu_0 c} \mathbf{E} \times (\mathbf{n} \times \mathbf{E}) = \frac{1}{\mu_0 c} [(\mathbf{E} \cdot \mathbf{E})\, \mathbf{n} - (\mathbf{E} \cdot \mathbf{n})\, \mathbf{E}]$$

$$= \frac{1}{\mu_0 c} E^2 \mathbf{n}$$

(since $\mathbf{E} \cdot \mathbf{n} = 0$, **E** being prependicular to **n**)

$$= \frac{E^2}{Z_0} \mathbf{n} \qquad \text{[refer equation (21)]}$$

For a plane electromagnetic wave of angular frequency ω, the average value of **S** over a complete cycle is given by

$$\langle \mathbf{S} \rangle = \frac{1}{Z_0} \langle E^2 \rangle \mathbf{n}$$

$$= \frac{1}{Z_0} \left\langle \left(E_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \right)^2 \right\rangle_{real} \mathbf{n}$$

$$= \frac{1}{Z_0} E_0^2 \langle \cos^2 (\omega t - \mathbf{k} \cdot \mathbf{r}) \rangle \mathbf{n}$$

$$= \frac{1}{Z_0} \frac{E_0^2}{2} \mathbf{n} \qquad \left[\text{since } (\cos^2 \langle \omega t - \mathbf{k} \cdot \mathbf{r}) \rangle = \tfrac{1}{2}\right]$$

$$= \frac{1}{Z_0} E_{rms}^2 \, \mathbf{n} \qquad \text{...(22)}$$

$$\left(\text{since } E_{rms} = \frac{E_0}{\sqrt{2}} \right)$$

It is obvious that the direction of Poynting vector is along the direction of propagation of electromagnetic wave. This means that the *flow of energy in a plane electromagnetic wave in free space is along the direction of wave.*

Ratio of Electrostatic and magnetic energy densities is given by

$$\frac{u_0}{u_m} = \frac{\frac{1}{2}\varepsilon_0 E^2}{\frac{1}{2}\mu_0 H^2} = \frac{\varepsilon_0}{\mu_0}\frac{E^2}{H^2} = \frac{\varepsilon_0}{\mu_0}\cdot\frac{\mu_0}{\varepsilon_0} = 1. \qquad \text{...(23)}$$

$$\left[\text{since } \frac{E}{H} = \sqrt{\left(\frac{\mu_0}{\varepsilon_0}\right)} \right]$$

i.e. the electromagnetic energy density is equal to magnetostatic energy density. Total *electromagnetic energy density*

$$\therefore \qquad u = u_e + u_m = 2u_e = 2 \times \frac{1}{2}\varepsilon_0 E^2 = \varepsilon_0 E^2$$

∴ Time average of *energy density*

$$< u > = < \varepsilon_0 E^2 > = \varepsilon_0 < (E_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t})^2 >_{real}$$

$$= \varepsilon_0 E_0^2 < \cos^2(\omega t - \mathbf{k}\cdot\mathbf{r}) > = \frac{1}{2}\varepsilon_0 E_0^2 = \varepsilon_0 E_{rms}^2 \qquad \text{...(24)}$$

Dividing (22) by (24), we obtain

$$\frac{<\mathbf{S}>}{<u>} = \frac{1}{Z_0 \varepsilon_0}\mathbf{n} = \frac{1}{\sqrt{\left(\frac{\mu_0}{\varepsilon_0}\right)}\,\varepsilon_0}\mathbf{n} = \frac{n}{\sqrt{\mu_0 \varepsilon_0}} = c\,\mathbf{n} \qquad \text{...(25)}$$

Thus we obtain

$$<\mathbf{S}> = <u> c\,\mathbf{n} \qquad \text{...(26a)}$$

i.e. energy flux = energy density × c. ...(26b)

This equation implies that the *energy density associated with an electromagnetic wave in free space propagates with the speed of light with which the field vectors do.*

Summarising we may say for electromagnetic waves in free space that :

1. In free space the electromagnetic waves travel with the speed of light.

2. The electromagnetic field vectors **E** and **H** are mutually perpendicular and they are also perpendicular to the direction of propagation of electromagnetic waves. Thereby indicating the electromagnetic waves are transverse in nature.

3. The field vectors **E** and **H** are in same phase.

4. The direction of flow of electromagnetic energy is along the direction of wave propagation and the energy flow per unit area per second is represented by

$$<\mathbf{S}> = \frac{E_{rms}^2}{Z_0}\mathbf{n} = <u> c\,\mathbf{n}.$$

5. The electrostatic energy density is equal to the magnetic energy density and the energy density associated with the electromagnetic wave in free space propagates with the speed of light.

$$H = \frac{1000}{16\pi^2 E} = \frac{1000}{16\pi^2 \times 48.87} = 0.1297 \text{ amp-turn/m}.$$

8.14. Plane Electromagnetic Waves in a Non-conducting Isotropic medium. (*i.e.* Isotropic Dielectric)

A non-conducting medium which has same properties in all directions is called an isotropic dielectric. Maxwell's equations are

$$\left.\begin{aligned} \text{div } \mathbf{D} &= \nabla \cdot \mathbf{D} = \rho \\ \text{div } \mathbf{B} &= \nabla \cdot \mathbf{B} = 0 \\ \text{curl } \mathbf{E} &= \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \text{and} \quad \text{curl } \mathbf{H} &= \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \end{aligned}\right\} \quad ...(7)$$

In an *isotropic dielectric* (or non-conducting isotropic medium)

$$\mathbf{D} = \varepsilon \mathbf{E}, \mathbf{B} = \mu \mathbf{H}, \mathbf{J} = \sigma \mathbf{E} = 0 \text{ and } \rho = 0$$

Therefore Maxwell's equations in this case take the form

$$\left.\begin{aligned} \text{div } \mathbf{E} &= \nabla \cdot E = 0 & ...(a) \\ \text{div } \mathbf{H} &= \nabla \cdot \mathbf{H} = 0 & ...(b) \\ \text{and} \quad \text{curl } \mathbf{E} &= -\mu \frac{\partial \mathbf{H}}{\partial t} & ...(c) \\ \text{curl } \mathbf{H} &= \varepsilon \frac{\partial \mathbf{E}}{\partial t} & ...(d) \end{aligned}\right\}$$

Taking curl of equation (2c), we get

$$\text{curl curl } \mathbf{E} = -\mu \frac{\partial}{\partial t} (\text{curl } \mathbf{H})$$

Substituting curl **H** from (2d) in above equation

$$\text{curl curl } \mathbf{E} = -\mu \frac{\partial}{\partial t} \left(\varepsilon \frac{\partial \mathbf{E}}{\partial t} \right)$$

i.e.

$$\text{curl curl } \mathbf{E} = -\mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad ...(3)$$

Similarly if we take curl of (2*d*) and substitute curl **E** from (2c), we get

$$\text{curl curl } \mathbf{H} = -\mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} \quad ...(4)$$

using vector identity

$$\text{curl curl } \mathbf{A} = \text{grad div } \mathbf{A} - \nabla^2 \mathbf{A}$$

and keeping in mind equations (2a) and (2b) *i.e.*, div **E** = 0 and div **H** = 0 equations (3) and (4) give

$$\nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad ...(5)$$

and

$$\nabla^2 \mathbf{H} - \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \quad ...(6)$$

These equations are vector equations of identical form which means that each of the six components of **E** and **H** separately satisfies the same scalar wave equation of the form

$$\nabla^2 u - \mu\varepsilon \frac{\partial^2 u}{\partial t^2} = 0 \qquad ...(7)$$

where u is a scalar and can stand for any one of components of **E** and **H**. It is obvious that equation (6) resembles with the general wave equation

$$\nabla^2 u - \frac{1}{v^2} \cdot \frac{\partial^2 u}{\partial t^2} = 0 \qquad ...(8)$$

where v is the speed of wave.

This means that the field vector **E** and **H** are propagated in isotropic dielectric as waves with speed v given by

$$v = \frac{1}{\sqrt{(\mu\varepsilon)}} = \frac{1}{\sqrt{(K_m \mu_0 K_e \varepsilon_0)}}$$

where K_m is relative permeability of medium and K_e is relative permittivity (or dielectric constant) of the medium.

As $\frac{1}{\sqrt{\mu_0\varepsilon_0}} = c$, speed of electromagnetic waves in free space.

$$\therefore \qquad v = \frac{c}{\sqrt{K_m K_e}} \qquad ...(10)$$

Since $K_m > 1$ and $K_e > 1$: thereby indicating ***that the speed of electromagnetic waves is an isotropic dielectric is less than the speed of electromagnetic waves in free space.***

As $$n = \frac{c}{v} \quad i.e. \; v = \frac{c}{n} \qquad ...(11)$$

$\therefore$ Comparing (10) and (11) we note that the refractive index n *in this particular case is*

$$n = \sqrt{(K_m K_e)} \qquad ...(12)$$

For a non-magnetic material $K_m = 1$; therefore

$$n = \sqrt{K_e} \quad i.e. \quad n^2 = K_e \qquad ...(13)$$

This relation is known as Maxwell's relation and has been verified by a number of experiments.

Replacing $\mu\varepsilon$ by $\frac{1}{v^2}$, wave equations (5) and (6) may be expressed as

$$\nabla^2 \mathbf{E} - \frac{1}{v^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \qquad ...(14)$$

and

$$\nabla^2 \mathbf{H} - \frac{1}{v^2}\frac{\partial^2 \mathbf{H}}{\partial t^2} = 0. \qquad ...(15)$$

The *plane-wave solutions* of equations (14) and (15) in well known from may be written as

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \qquad ...(16)$$

$$\mathbf{H}(\mathbf{r}, t) = \mathbf{H}_0 \, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \qquad ...(17)$$

where $\mathbf{E}_0$ and $\mathbf{H}_0$ are complex amplitudes which are constant in space and time : while **k** is wave propagation vector given by

$$\mathbf{k} = k\hat{n} = \frac{2\pi}{\lambda}\hat{n} = \frac{\omega}{v}\hat{n} \quad ...(18)$$

Here $\hat{n}$ is a unit vector in the direction of wave propagation.

Relative directions of E and H. The requirement $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{H} = 0$, demand that

$$\mathbf{k} \cdot \mathbf{E} = 0 \quad \text{and} \quad \mathbf{k} \cdot \mathbf{H} = 0 \quad ...(19)$$

Comparing (7) and (8), we see

$$v = \frac{1}{\sqrt{(\mu\varepsilon)}} \quad ...(19a)$$

This means that the field vector **E** and **H** are both perpendicular to the direction of propagation vector **k**. This implies that ***electromagnetic waves in isotropic dielectric are transverse in nature*** Further restrictions are provided by curl equations (2c) and (2b) *viz.*

$$\text{curl } \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad \text{and} \quad \text{curl } \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

Using (16) and (17), these equations yield

$$\mathbf{k} \times \mathbf{E} = \mu\omega\mathbf{H} \quad ...(20)$$

and

$$\mathbf{k} \times \mathbf{H} = -\varepsilon\omega\mathbf{E} \quad ...(21)$$

From these equations it is obvious that *field vectors* **E** *and* **H** *are mutually perpendicular and also they are perpendicular to the direction of propagation vector* **k**. *This in turn implies that in a plane electromagnetic wave in isotropic dielectric, vector* **(E, H, k)** *from a set of orthogonal vectors which form a right handed coordinate system in given order (fig. 8.2).*

Phase of E and H and Wave Impedance. From equation (20)

$$\mathbf{H} = \frac{1}{\mu\omega}(\mathbf{k} \times \mathbf{E}) = \frac{k}{\mu\omega}(\hat{n} \times \mathbf{E})$$

$$= \frac{1}{\mu v}(\hat{n} \times \mathbf{E}) = \sqrt{\frac{\varepsilon}{\mu}}(\hat{n} \times \mathbf{E}) \quad ...(22)$$

$$\left(\text{since } k = \frac{\omega}{v} \text{ and } v = \frac{1}{\sqrt{(\mu\varepsilon)}}\right)$$

Now the ratio of magnitude of **E** to the magnitude of **H** is symbolised by Z *i.e.*

$$Z = \left|\frac{\mathbf{E}}{\mathbf{H}}\right| = \frac{\mathbf{E}_0}{\mathbf{H}_0} = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\left(\frac{K_m\mu_0}{K_e\varepsilon_0}\right)} = \text{real quantity} \quad ...(23)$$

This implies ***that the field vectors* E *and* H *are in the same phase, i.e., they have same relative magnitudes at all points at all time. The unit of Z comes out to be ohm, since***

$$\mathbf{Z} = \frac{\mathbf{E}}{\mathbf{H}} = \frac{\text{volt/m}}{\text{amp–turn/m}} = \frac{\text{volt}}{\text{amp}} = \text{ohm};$$

hence the value of Z is referred to as *wave impedance of isotropic dielectric medium.* The *wave impedance of medium is related to that of free space by the relation*

$$Z = \sqrt{\left(\frac{K_m\mu_0}{K_e\varepsilon_0}\right)} = \sqrt{\left(\frac{K_m}{K_e}Z_0\right)} \quad ...(24)$$

where $Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$ is called the wave impedance of free space.

Poynting vector for a plane electromagnetic wave in an isotropic dielectric is given by

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \mathbf{E} \times \left(\sqrt{\frac{\varepsilon}{\mu}}\,\hat{n} \times \mathbf{E}\right)$$

$$= \frac{\mathbf{E} \times (\hat{n} \times \mathbf{E})}{Z} \quad \left(\text{since } Z = \sqrt{\frac{\mu}{\varepsilon}} \right)$$

$$= \frac{(\mathbf{E} \cdot \mathbf{E})\, \hat{n} - (\mathbf{E} \cdot \hat{n})\, \mathbf{E}}{\mathbf{Z}}$$

$$= \frac{\mathbf{E}^2}{Z} \hat{n} \quad (\text{since } \mathbf{E} \cdot \hat{n} = 0 \text{ because } \mathbf{E} \text{ is perpendicular to } \hat{n})$$

The time average of *Poynting vector* is

$$<\mathbf{S}> = <\mathbf{E} \times \mathbf{H}> = <\frac{\mathbf{E}^2}{Z} \hat{n}>$$

$$= \frac{1}{Z} <\left(\mathbf{E}_0 \, e^{i \mathbf{k} \cdot \mathbf{r} - i\omega t} \right)^2>_{real} \hat{n}$$

Since for finding actual physical fields we often take real parts of complex exponentials.

$$\therefore \quad <\mathbf{S}> = \frac{1}{Z} E_0^2 <\cos^2 (\omega t - \mathbf{k}\ \mathbf{r})> \hat{n}$$

$$= \frac{1}{Z} E_0^2 \cdot \frac{1}{2} \hat{n} = \frac{1}{2Z} E_0^2 \hat{n}$$

$$= \frac{1}{Z} E_{rms}^2 \hat{n} \quad \left(\text{since } E_{rms} = \frac{E_0}{\sqrt{2}} \right) \qquad \text{...(25a)}$$

$$= \sqrt{\left(\frac{K_e}{K_m} \right)} \frac{1}{Z_0} E_{rms}^2 \hat{n}$$

$$= \frac{\sqrt{(K_e K_m)}}{K_m} \frac{1}{Z_0} E_{rms}^2 \hat{n} = \frac{n}{K_m} \frac{1}{Z_0} E_{rms}^2 \hat{n}. \qquad \text{...(25b)}$$

[becuasé refractive index $n = \sqrt{(K_e K_{m)}}$]

$$= \frac{n}{K_m} <\mathbf{S}>_{\text{free space}} \qquad \text{...(25c)}$$

$$\left[\text{because } <\mathbf{S}>_{\text{free space}} = \frac{1}{Z_0} E_{rms}^2 \hat{n} \right]$$

Equations (25a) and (25b), show that the *flow of energy is along the direction of propagation of electromagnetic wave.* Equation (25c) shows that the Poynting vector for electromagnetic wave in isotropic dielectric is $\sqrt{\left(\frac{K_e}{K_m}\right)}$ or $\frac{n}{K_m}$ times of the Poynting vector if the same electromagnetic wave were propagated through free space. It may be noted that the average of Poynting vector may also be obtained as

$$<\mathbf{S}> = <\mathbf{E} \times \mathbf{H}> = \frac{1}{2} \cdot \text{Real part of } (\mathbf{E} \times \mathbf{H}^*)$$

Power flow and Energy density. Let us find the ratio of electrostatic and magnetostatic energy densities in an electromagnetic wave field *i.e.*

$$\frac{u_e}{u_m} = \frac{\frac{1}{2} \varepsilon E^2}{\frac{1}{2} \mu H^2} = \frac{\varepsilon}{\mu} \frac{E^2}{H^2} = \frac{\varepsilon}{\mu} Z^2 = \frac{\varepsilon}{\mu} \cdot \frac{\mu}{\varepsilon} = 1. \qquad \text{...(26)}$$

$$\left[\text{since } Z = \frac{E}{H} = \sqrt{\left(\frac{\mu}{\varepsilon} \right)} \right]$$

This implies that for the case of electromagnetic waves in an isotropic dielectric the electrostatic energy density (u_e) is equal to the magnetostatic energy density (u_m).

Therefore total *electromagnetic energy density*

$$u = u_e + u_m = 2u_e \quad (\text{since } u_e = u_m)$$
$$= 2 \cdot \frac{1}{2}\varepsilon E^2 = \varepsilon E^2$$

Therefore *time average of energy density*

$$< u > = < \varepsilon E^2 > = \varepsilon < E^2 > = \varepsilon < \left(E_0 \, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \right)^2_{real} >$$

$$= \varepsilon E^2 < \cos^2 (\omega t - \mathbf{k}\cdot\mathbf{r} > = \frac{\varepsilon E_0^2}{2}$$

$$= \varepsilon E_{rms}^2 \quad ...(27)$$

Dividing equaion (23a) with equation (27), we obtain

$$\frac{< \mathbf{S} >}{< \mu >} = \frac{(E_{rms}^2 \hat{n}/Z)}{\varepsilon E_{rms}^2} = \frac{1}{Z\varepsilon}\hat{n} = \frac{2}{\sqrt{\left(\frac{\mu}{\varepsilon}\right)}\,\varepsilon}\hat{n} \text{ since } Z = \sqrt{\frac{\mu}{\varepsilon}}$$

$$= \frac{1}{\sqrt{\mu\varepsilon}}\hat{n} = v\hat{n} \quad \left(\text{since } v = \frac{1}{\sqrt{\mu\varepsilon}}\right)$$

Thus we obtain

$$< \mathbf{S} > = < u > v\hat{n} \quad ...(28a)$$

or

$$< \mathbf{S} >_{av} = u_{av} \, v\hat{n} \quad ...(28b)$$

or in words **energy flux = v × energy density.**

This equation has a simple meaning. If the energy were flowing with velocity v (= phase velocity of electromagnetic wave with which electromagnetic field vectors propagate), in the direction of propagation of wave, all the energy contained in a cylinder of unit cross-section and height equal to v would cross unit cross-section per second, forming the flux. This in turn implies *that the energy density associated with an electromagnetic wave in a stationary homogeneous nonconducting medium propagates with the same speed with which the field vectors do.*

Summarising we may say for the case of electromagnetic waves in isotropic dielectric that :

1. In isotropic dielectric the electromagnetic waves travel with a speed less than the speed of light.

2. The electromagnetic field vectors **E** and **H** are mutually perpendicular and they are also perpendicular to the direction of propagation of electromagnetic wave. Thereby indicating that electromagnetic waves are transverse in nature.

3. The field vectors **E** and **H** are in the same phase.

4. The direction of flow of electromagnetic energy is along the direction of wave propagation and the energy flow per unit area per second is represented as

$$< S > = \frac{E_{rms}^2 \hat{n}}{Z} = < u > v\hat{n}$$

5. The electrostatic energy density is equal to the magnetostatic energy density and the total energy density is given by

$$< u > = \varepsilon E_{rms}^2 .$$

8.15. Plane Electromagnetic Waves in a Conducting Medium.

Maxwell's equations are

$$\left.\begin{aligned} \text{div}\,\mathbf{D} &= \nabla\cdot\mathbf{D} = \rho \\ \text{div}\,\mathbf{B} &= \nabla\cdot\mathbf{B} = 0 \\ \text{curl}\,\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t} \\ \text{curl}\,\mathbf{H} &= \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t} \end{aligned}\right\} \qquad ...(1)$$

Let us assume that medium is linear and isotropic and is characterised by permitivity ε, permeability μ and conductivity σ, but not any charge or any current other than that determined by Ohm's law. Then

$$\mathbf{D} = \varepsilon\mathbf{E},\ \mathbf{B} = \mu\mathbf{H},\ \mathbf{J} = \sigma\mathbf{E} \quad \text{and} \quad \rho = 0.$$

So that Maxwell's equation (1) in this case take the form

$$\left.\begin{aligned} \text{div}\,\mathbf{E} &= 0 && ...(a) \\ \text{div}\,\mathbf{H} &= 0 && ...(b) \\ \text{curl}\,\mathbf{E} &= -\mu\frac{\partial\mathbf{H}}{\partial t} && ...(c) \\ \text{curl}\,\mathbf{H} &= \sigma\mathbf{E} + \varepsilon\frac{\partial\mathbf{E}}{\partial t} && ...(d) \end{aligned}\right\} \qquad ...(2)$$

Taking curl of equation [2(c)], we get

$$\text{curl curl}\,\mathbf{E} = -\mu\frac{\partial}{\partial t}(\text{curl}\,\mathbf{H})$$

Substituting curl **H** from [2(d)], we get

$$\text{curl curl}\,\mathbf{E} = -\mu\frac{\partial}{\partial t}\left(\sigma\mathbf{E} + \varepsilon\frac{\partial\mathbf{E}}{\partial t}\right)$$

i.e.

$$\text{curl curl}\,\mathbf{E} = -\sigma\mu\frac{\partial\mathbf{E}}{\partial t} - \varepsilon\mu\frac{\partial^2\mathbf{E}}{\partial t^2} \qquad ...(3)$$

Similarly, if we take the curl of [2 (d)] and substitute curl **E** from [2 (c)], we obtain

$$\text{curl curl}\,\mathbf{H} = -\sigma\mu\frac{\partial\mathbf{H}}{\partial t} - \varepsilon\mu\frac{\partial^2\mathbf{H}}{\partial t^2} \qquad ...(4)$$

Now using vector identity

$$\text{curl curl}\,\mathbf{A} = \text{grad div}\,\mathbf{A} - \nabla^2\mathbf{A}$$

and keeping in view equations [2(a)] and [2(b)] (*i.e.* div **E** = 0 and div **H** = 0), equations (3) and (4) take the form

$$\nabla^2\mathbf{E} - \sigma\mu\frac{\partial\mathbf{E}}{\partial t} - \varepsilon\mu\frac{\partial^2\mathbf{E}}{\partial t^2} = 0 \qquad ...(5)$$

$$\nabla^2 \mathbf{H} - \sigma\mu \frac{\partial \mathbf{H}}{\partial t} - \varepsilon\mu \frac{\partial^2 \mathbf{H}}{\partial t} = 0 \quad ...(6)$$

These equations represent wave equaitons governing electromagnetic fields **E** and **H** in a homogeneous isotropic conducting medium of conductivity σ. It is apparent that these equations are vector equations of identical form; which means that each of the six components of **E** and **H** separately satisfies the same scalar wave equation of the form

$$\nabla^2 \psi - \sigma\mu \frac{\partial \psi}{\partial t} - \varepsilon\mu \frac{\partial^2 \psi}{\partial t} = 0 \quad ...(7)$$

where ψ is a scalar and can stand for any one of components of **E** and **H**.

In an *isotropic dielectric* we have seen that the *time varying fields are transverse i.e.* the field vectors **E** and **H** are perpendicular to the direction in which the spatial variation occurs. In the limit of zero frequency we know from electrostatics and magnetostatics that the *static fields in a dielectric are longitudinal* in the sense that the fields are derivable from scalar potentials and so point in the direction of spatial variation.

If the conductivity is not zero, modifications are necessary. For simplicity let us assume that the fields vary in only one spatial variable x_α. Therefore decomposing the fields into longitudinal and transverse parts :

$$\left.\begin{aligned} \mathbf{E}(x_\alpha, t) &= \mathbf{E}_l(x_\alpha, t) + \mathbf{E}_t(x_\alpha, t) \\ \mathbf{H}(x_\alpha, t) &= \mathbf{H}_l(x_\alpha, t) + \mathbf{H}_t(x_\alpha, t) \end{aligned}\right\} \quad ...(8)$$

where subscript l and t denoted longitudinal and transverse parts respectively. Then, because of properties of curl operation, we find that the transverse parts of **E** and **H** satisfy the two curl equations [2(c)] and (2(b)] ; leading to transverse wave, while the longitudinal parts satisfy the equations :

From equation [2(a)] $\quad \dfrac{\partial \mathbf{E}_l}{\partial x_\alpha} = 0 \quad$ ...[9(a)]

From equation [2(b)] $\quad \dfrac{\partial \mathbf{H}_l}{\partial x_\alpha} = 0 \quad$ ...[9(b)]

From equation [2(c)] $\quad \left(\dfrac{\partial}{\partial t} + \dfrac{\sigma}{\varepsilon}\right) \mathbf{E}_l = 0 \quad$ ...[9(c)]

[since curl $\mathbf{E}_l$ = curl grad $\phi_l = 0$]

From equation [2(d)] $\quad \dfrac{\partial \mathbf{H}_l}{\partial t} = 0 \quad$ ...[9(d)]

[since curl $\mathbf{H}_l$ = curl grad $\phi_l = 0$]

Equations [9(c)], and [9(d)], show that the only longitudinal magnetic field is possible in a static uniform field. This is the same situation as the case of a dielectric. But equations 9(a) and 9(c) show that the longitudinal electric field is uniform in space, while possesses the *time variation* given by

$$\left(\frac{\partial}{\partial t} + \frac{\sigma}{\varepsilon}\right) E_l = 0 \quad i.e. \quad \frac{\partial E_l}{\partial_l} = -\frac{\sigma}{\varepsilon} E_l$$

i.e.

$$\frac{\partial E_l}{E_l} = -\frac{\sigma}{\varepsilon} dt$$

Integrating, we get

$$\log E_l = -\frac{\sigma}{\varepsilon} t + \log E_0 \quad \text{(where } \log E_0 \text{ is constant of integration)}$$

i.e.

$$E_l(\alpha, t) = E_0 \, e^{-(\sigma/\varepsilon) i} \quad ...(10)$$

Consequently, *no static electric fields can exist in a conducting medium in the absence of an applied current density*. For good conductors like copper ; $\sigma \approx 10^7$ mho/m, so that disturbances are damped out in an extremely short time.

Therefore we shall consider the transverse field in the conducting medium. Let us assume that the fields vary as $e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$, then solutions of equations (5), (6) and (7) may expressed as

$$\mathbf{E} = \mathbf{E}_0\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \qquad ...(11)$$

$$\mathbf{H} = \mathbf{H}_0\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \qquad ...(12)$$

$$\psi = \psi_0\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \qquad ...(13)$$

where **k** is a wave vector, may be complex, while $\mathbf{E}_0, \mathbf{H}_0$ and ψ_0 are complex amplitudes which are constant in space and time.

Substituting value of ψ from (13) in eqn. (7), we obtain

$$(-k^2 + i\sigma\mu\omega + \mu\varepsilon\omega^2)\,\psi = 0.$$

Since ψ is arbitrary, therefore this equation holds only if

$$(-k^2 + i\sigma\mu\omega + \mu\varepsilon\omega^2) = 0.$$

This means that the *propagation wave vector* **k** is complex given by

$$k^2 = \mu\varepsilon\omega^2\left(1 + \frac{i\sigma}{\omega\varepsilon}\right) \qquad ...(14)$$

In above equation the first term corresponds to the displacement current and the second to the conduction current contribution. As k is complex we may write assuming that σ is real.

$$k = \alpha + i\beta \qquad ...(15)$$

So that

$$k^2 = \alpha^2 - \beta^2 + 2i\alpha\beta. \qquad ...(16)$$

Comparing equation (14) and (16), we get

$$\left.\begin{aligned} \alpha^2 - \beta^2 &= \mu\varepsilon\omega^2 \\ \text{and} \quad 2\alpha\beta &= \mu\omega\sigma \end{aligned}\right\} \qquad ...(17)$$

$$\left.\begin{aligned} \alpha &= \sqrt{\mu\varepsilon}\cdot\omega\left[\frac{\sqrt{\left\{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right\}} + 1}{2}\right]^{1/2} \\ \beta &= \sqrt{\mu\varepsilon}\cdot\omega\left[\frac{\sqrt{\left\{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right\}} - 1}{2}\right]^{1/2} \end{aligned}\right\} \qquad ...(18)$$

Now in terms of α and β, the field vectors **E** and **H** take the form

$$\mathbf{E} = \mathbf{E}_0\, e^{i(\alpha + i\beta)\,\mathbf{n}\cdot\mathbf{r} - i\omega t} = \mathbf{E}_0\, e^{-\beta\,\mathbf{n}\cdot\mathbf{r}}\, e^{i(\alpha\mathbf{n}\cdot\mathbf{r} - \omega t)} \qquad ...(19)$$

and

$$\mathbf{H} = \mathbf{H}_0\, e^{i(\alpha + i\beta)\,\mathbf{n}\cdot\mathbf{r} - i\omega t} = \mathbf{H}_0\, e^{-\beta\,\mathbf{n}\cdot\mathbf{r}}\, e^{i(\alpha\mathbf{n}\cdot\mathbf{r} - \omega t)} \qquad ...(20)$$

From equations (19) and (20) it is obvious that *field amplitudes are spatially attenuated due to the presence of term* $e^{-\beta\,\mathbf{n}\cdot\mathbf{r}}$. The quantity β is a measure of attenuation and is known as absorption coefficient. Also in last exponential terms in (19) and (20) the usual notation **k** has been replaced by α, therefore we conclude that the field vectors are propagated in the conducting medium with speed ($v = \omega/k$) given by

$$v = \frac{\omega}{\alpha} = \frac{1}{\sqrt{\mu\varepsilon}}\left[\frac{\sqrt{\left\{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right\}}+1}{2}\right]^{-1/2} \quad \text{using (18)} \quad ...(21)$$

Now let us consider the form of propagation vector $k = \alpha + i\beta$ in the two particular cases :

Case (i) *For a poor conductor* $\frac{\sigma}{\omega} \ll 1$, then we get

$$\alpha = \sqrt{\mu\varepsilon\omega} \text{ and } \beta = \sqrt{\frac{\mu}{\varepsilon}} \cdot \frac{\sigma}{2}$$

$$\therefore \quad k = \alpha + i\beta = \sqrt{\mu\varepsilon\omega} + i\,\frac{\sigma}{\varepsilon}\sqrt{\frac{\mu}{\varepsilon}} \quad ...(22)$$

This is correct to first order in $\sigma/\omega\varepsilon$. In this limit $\alpha \gg \beta$ and the attenuation of wave determined by β is independent of frequency, aside from the possible variation of conductivity.

Case (ii) *For a good conductor* $\sigma/\omega\varepsilon \gg 1$, so that α and β are approximately equal *i.e.*

$$\alpha \approx \beta = \mu\varepsilon \cdot \omega\sqrt{\frac{(\sigma/\omega\varepsilon)}{2}} = \sqrt{\frac{\mu\sigma\omega}{2}} \quad ...(23)$$

$$\therefore \quad k \approx \alpha + i\beta = (1+i) = \sqrt{\frac{\mu\sigma\omega}{2}} \quad ...(24)$$

where only lowest order term in $\omega\varepsilon/\sigma$ have been kept.

Skin Depth or Penetration Depth. The waves given by equation (19) & (20) show an exponential damping or attenuation with distance. Greater is the value of β, greater is the attenuation. The term $1/\beta$ measures, the depth at which electromagnetic wave entering a conductor is damped to $1/e = 0{\cdot}369$. of its initial amplitude at the surface. This depth is known as the *skin depth* or the *penetration depth* and is usually represented by δ,

$$\delta = \frac{1}{\beta} = \frac{1}{\omega\sqrt{\mu\varepsilon}}\left[\frac{\sqrt{\left\{1+\left(\frac{\sigma}{\omega\varepsilon}\right)\right\}^2}-1}{2}\right]^{-1/2} \quad ...(25a)$$

$$= \sqrt{\frac{2}{\mu\sigma\omega}} \left[\text{for good conductors characterised by } \frac{\sigma}{\omega\varepsilon} \ll 1\right] \quad ...(25b)$$

It is obvious that the skin depth decreases with increasing frequency. For copper at 60 cycles δ is 0·86 cm, but at 1 megacycle, it has dropped to 0·0067. That is why in high frequency circuits current flows only on the surface of the conductors. The major importance of the skin depth is that it measures the depth to which an electromagnetic wave can penetrate a conducting medium. Therefore, the conducting sheets which are used as electromagnetic shields must be thicker than the skin depth.

Remarks (i) *For silver* $\sigma \approx 10^7$ *mho/m at a typical microwave frequency* $\approx 10^8$ *c/s, the skin depth* $\approx 10^{-4}$ *cm. Thus at microwave frequencies the skin depth in silver is very small and consequently performance of a pure silver component and a silver plated brass component would be expected to be indistinguishable.*

(ii) *For see water* $\sigma = 4{\cdot}3$ *mho/m at a frequency of 60 kc/s ; so that* $\delta \approx 1$ *meter. That is why radiocommunication with submerged submarine becomes increasing difficult at several skin depths.*

Relative directions of E and H. Substituting **E** and **H** from (11) and (12) in Maxwell's divergence equation, we obtain

$$i\,\mathbf{k}\cdot\mathbf{E} = 0 \quad \text{or} \quad \mathbf{k}\cdot\mathbf{E} = 0 \qquad \text{...(26)}$$

$$i\,\mathbf{k}\cdot\mathbf{H} = 0 \quad \text{or} \quad \mathbf{k}\cdot\mathbf{H} = 0 \qquad \text{...(27)}$$

These equations imply that field vectors **E** and **H** are both perpendicular to the direction of propagation vector **k**. This implies that electromagnetic wave in a conducting medium is transverse. Further restrictions on **E** and **H** is imposed by curl equations. Using (11) and (12) curl equations demand

$$i\,\mathbf{k}\times\mathbf{E} = i\mu\omega\mathbf{H} \quad i.e. \quad \mathbf{k}\times E = \mu\omega\,\mathbf{H} \qquad \text{...(28)}$$

and

$$i\,\mathbf{k}\times\mathbf{H} = (\sigma - i\varepsilon\omega)\,\mathbf{E} \quad i.e. \quad \mathbf{k}\times\mathbf{H} = -(\varepsilon\omega + i\sigma)\,\mathbf{E} \qquad \text{...(29)}$$

These equations imply that electromagnetic field vectors **E** and **H** are mutually perpendicular and also they are perpendicular to the direction of propagation vector **k.**

Phase of E and H. From equation (28)

$$\mathbf{H} = \frac{1}{\mu\omega}(\mathbf{k}\times\mathbf{E}) = \frac{1}{\mu\omega}k(\mathbf{n}\times\mathbf{E})$$

$$= \frac{1}{\mu\omega}(\alpha + i\beta)(\mathbf{n}\times\mathbf{E}) \qquad \text{...(29)}$$

This implies that

$$\left|\frac{\mathbf{H}}{\mathbf{E}}\right| = \frac{H_0}{E_0} = \frac{\alpha + i\beta}{\mu\omega} = \text{complex quantity} \qquad \text{...(30)}$$

i.e. field vectors **H** *and* **E** *are out of phase in a conductor.* The magnitude and phase of complex k, written as $k = |k|\,e^{i\phi}$, may be defined as

$$|k| = |\alpha + i\beta| = \sqrt{(\alpha^2+\beta^2)} \times \sqrt{\mu\varepsilon}\,\omega\left[1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/2} \qquad \text{...(31)}$$

and

$$\phi = \tan^{-1}\left(\frac{\beta}{\alpha}\right) = \frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\varepsilon}\right) \qquad \text{...(32)}$$

so equation (29) may be expressed as

$$\mathbf{H} = \frac{1}{\mu\omega}\,\omega\sqrt{\mu\varepsilon}\left[1 + \left(\frac{\sigma}{\mu\omega}\right)^2\right]^{1/4} e^{i\phi}\,(\mathbf{n}\times\mathbf{E})$$

$$= \sqrt{\frac{\varepsilon}{\mu}}\left[1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/2} e^{-i(-\phi)}\,(\mathbf{n}\times\mathbf{E}) \qquad \text{...(33)}$$

This interpretation of this equation is that **H** lags behind **E** in time by the phase angle ϕ given by equation (32) and has a relative magnitude

$$\left|\frac{\mathbf{H}}{\mathbf{E}}\right| = \frac{H_0}{E_0} = \sqrt{\frac{\varepsilon}{\mu}}\left[1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4} \qquad \text{...(34)}$$

Poynting Vector. The Poynting vector is given by

$$\mathbf{S} = (\mathbf{E}\times\mathbf{H})$$

the *time average of Poynting vector* may expressed as

$$S_{av} = \frac{1}{2}\,\text{Real part of }(\mathbf{E}\times H^*) = \frac{1}{2}Re\,(\mathbf{E}\times\mathbf{H}^*)$$

where $\mathbf{H}^*$ (denotes complex conjugate of **H** and *Re* denotes *real part of.*

$$\mathbf{S}_{av} = \frac{1}{2}Re\left[\mathbf{E}\times\left\{\sqrt{\frac{\varepsilon}{\mu}}\left(1 + \frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/2} e^{-i\phi}\,\mathbf{n}\times\mathbf{E}^*\right\}\Bigg]$$

$$= \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\left[1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4} Re\,\{\mathbf{E}\times(\mathbf{n}\times\mathbf{E}^*)\,e^{-i\phi}\}$$

$$= \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\left[1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4} Re\ [\{(\mathbf{E}\cdot\mathbf{E}^*)\,\mathbf{n}) - (\mathbf{E}\cdot\mathbf{n})\,\mathbf{E}^*\}\,e^{-i\phi}]$$

$$= \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\left[\left(1+\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4} E_0^2\, e^{-2\beta\,\mathbf{n}\cdot\mathbf{r}}\ \mathbf{n}\cos\phi \qquad ...(35)$$

$$\left[\text{Since } (\mathbf{E}\cdot\mathbf{E}^*) = E_0^2\, e^{-2\beta\,\mathbf{n}\cdot\mathbf{r}} \text{ and } Re\,(e^{-i\phi}) = \cos\phi\right]$$

For good conductors $\sigma/\varepsilon\omega \gg 1$ so that $\phi = \pi/4$ and also $E_{rms} = \frac{E_0}{\sqrt{2}}$ hence.

$$\mathbf{S}_{av} = \sqrt{\left\{\left(\frac{\sigma}{2\mu\omega}\right)\right\}} E_{rms}^2\, e^{-2\beta\,\mathbf{n}\cdot\mathbf{r}}\ \mathbf{n}$$

Energy density. The total energy density of electromagnetic field is given by

$$u = u_e + u_m$$

where electrostatic energy

$$u_e = \frac{1}{2} Re \frac{1}{2}\,(\mathbf{E}\cdot\mathbf{D}^*)$$

$$= \frac{1}{4}\varepsilon\, Re\,(\mathbf{E}\cdot E^*)$$

$$= \frac{1}{4}\,\varepsilon E_0^2\, e^{-2\beta\,\mathbf{n}\cdot\mathbf{r}} \qquad ..(37a)$$

$$= \frac{1}{2}\,\varepsilon E_{rms}^2\, e^{-2\beta\,\mathbf{n}\cdot\mathbf{r}} \qquad ...(37b)$$

and magnetic energy density

$$u_m = \frac{1}{2}\ Re\ \frac{1}{2}\,(\mathbf{H}\cdot\mathbf{B}^*)$$

$$= \frac{1}{4}\,\mu\ Re\ (\mathbf{H}\cdot\mathbf{H}^*)$$

$$= \frac{1}{4}\,\mu\ H_0^2\ e^{-2\beta\,\mathbf{n}\cdot\mathbf{r}}$$

$$= \frac{1}{4}\,\mu\,\frac{\varepsilon}{\mu}\left\{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right\}^{1/2} E_0^2\, e^{-2\beta\,\mathbf{n}\cdot\mathbf{r}}$$

[using (34)]

$$= \frac{1}{4}\,\varepsilon\left\{1+\left(\frac{\sigma}{\varepsilon\,\omega}\right)^2\right\}^{1/2} E_0^2\, e^{-2\beta\,\mathbf{n}\cdot\mathbf{r}}$$

$$= \left[1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2\right]^{1/2} u_e \qquad ...(38)$$

[using (37a)]

$$\therefore\ \text{Total energy density } u = u_e + u_m = u_e + \left\{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2\right\}^{1/2} u_e$$

$$= \left[1+\left\{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2\right\}\right]^{1/2} u_e$$

$$= \left[1+\left\{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2\right\}\right]^{1/2} \times \frac{1}{2}\,\varepsilon E_{rms}^2\ e^{-2\beta\,\mathbf{n}\cdot\mathbf{r}} \qquad ...(39)$$

From equations (36) and (39) it is obvious that the ***energy flux and energy density are damped as the electromagnetic wave propagates in a conducting medium.*** This energy lose is due to Joule heating of the medium. From equations (37) and (38) it is also obvious that ***in a conducting medium the electrostatic and magnetic energy densities are different ; the magnetic energy density being greater than electrostatic energy density.***

9.2. Reflection and Refraction of Electromagnetic Waves at the Interface of Non-conducting Media.

The reflection and refraction of light at a plane surface between two media of different dielectric properties are familiar phenomenon. The various aspects of the phenomenon divide themselves into two categories :

1. Kinematic Properties.

(i) Law of reflection. The angle of reflection is equal to the angle of incidence *i.e.* $\theta_r' = \theta_i$ where θ_i is angle of incidence and θ'_r is angle of reflection.

(ii) Snell's law of refraction states.

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{n_2}{n_1}$$

where θ_i and θ_r are the angles of incidence and refraction ; while n_1 and n_2 are corresponding indices of refraction.

(iii) Law of frequency. The incident, reflected and refracted waves all have the same frequency.

2. Dynamic Properties. These properties are concerned with :

(i) intensities of reflected and refracted waves.

(ii) phase changes and polarisation.

The kinematic properties follow immediately from the wave nature of the phenomenon and the fact that there are certain boundary conditions imposed on field vectors to be satisfied. But they do not depend on the nature of waves or the boundary conditions. On the other hand, the dynamic properties depend entirely on the specific nature of electromagnetic field and their boundary conditions.

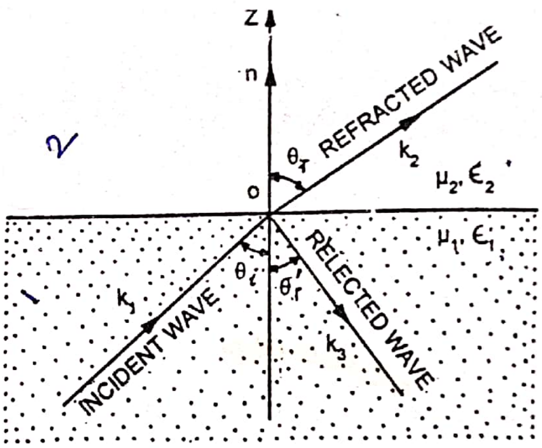

Fig. 9.3 Reflection and refraction

Let us consider a plane interface at $Z = 0$ separating two homogeneous, chargefree and non-conducting isotropic media, characterised by permittivities ε_1 and ε_2 ; permeabilities μ_1 and μ_2 respectively as shown in fig. 9.3. Let a plane wave with wave vector $\mathbf{k}_1$ and frequency ω_1 be incident from medium '*I*' at point O on the interface. This wave is partly reflected and partly transmitted (or refracted). Let the reflected and refracted waves have wave vectors $\mathbf{k}'_1$ and $\mathbf{k}_2$, frequencies ω'_1 and ω_2 respectively. Also let $\mathbf{n}$ be the unit vector normal to the interface and directed from medium '1' into medium '2'. The field vectors for incident reflected and refracted waves may be expressed as

$$\left.\begin{aligned} \textit{For incident wave}\quad \mathbf{E}_1 &= \mathbf{E}_{01}\, e^{i\mathbf{k}_1\cdot\mathbf{r} - i\omega_1 t} \\ \mathbf{B}_1 &= \frac{\mathbf{k}_1 \times \mathbf{E}_1}{\omega_1} \quad \text{or} \quad \mathbf{H}_1 = \frac{\mathbf{k}_1 \times \mathbf{E}_1}{\mu_1\omega_1} \end{aligned}\right\} \quad \text{...(1)}$$

$$\left.\begin{aligned} \textit{For reflected wave}\quad \mathbf{E}'_1 &= \mathbf{E}'_{01}\, e^{i\mathbf{k}_1'\cdot\mathbf{r} - i\omega t} \\ \mathbf{B}'_1 &= \frac{\mathbf{k}_1' \times \mathbf{E}_1'}{\omega_1'} \quad \text{or} \quad \mathbf{H}_1' = \frac{\mathbf{k}_1' \times \mathbf{E}_1'}{\mu_1\omega_1'} \end{aligned}\right\} \quad \text{...(2)}$$

$$\left.\begin{aligned} \textit{For refracted wave}\quad \mathbf{E}_2 &= \mathbf{E}_{02}\, e^{i\mathbf{k}_2\cdot\mathbf{r} - i\omega t} \\ \mathbf{B}_2 &= \frac{\mathbf{k}_2 \times \mathbf{E}_2}{\omega_2} \quad \text{or} \quad \mathbf{H}_2 = \frac{\mathbf{k}_2 \times \mathbf{E}_2}{\mu_2\omega_2} \end{aligned}\right\} \quad \text{...(3)}$$

Note. The cause of notation used above may be understood as : Incident and reflected wave are in medium 'I', hence incident wave has been characterised by symbol I (unprimed), while reflected wave has been represented by symbol I (primed). The refracted wave is in medium '2', hence it has been represented by symbol 2.

We can apply the boundary condition (iii) derived in preceding section namely that the tangential component of the electric field is continuous across the interface between the media (*i.e.* at $z = 0$). In this case at every point in the interface

$$(E_1)_{tangential} + (E_1)'_{tangential} = (E_2)_{tangential} \qquad ...(4)$$

or

$$(E_{01})_{tangential}\, e^{i\,\mathbf{k}_1 \cdot \mathbf{r} - i\omega t} + (E_{01}')_{tangential}\, e^{i\,\mathbf{k}_1' \cdot \mathbf{r} - i\omega_1' t}$$

$$= (E_{02})_{tangential}\, e^{i\,\mathbf{k}_2 \cdot \mathbf{r}}\, e^{-i\omega_2 t}$$

i.e.

$$(E_{01})_{tangential}\, e^{i\,\mathbf{k}_1 \cdot \mathbf{r}}\, e^{-i\omega t} + (E_{01}')_{tangential}\, e^{i\,\mathbf{k}_1 \cdot \mathbf{r}}\, e^{i\omega_1' t}$$

$$= (E_{02})_{tangential}\, e^{i\,\mathbf{k}_2 \cdot \mathbf{r}}\, e^{-i\omega_2 t} \qquad ...(5)$$

Since this equality is independent of time, it immediately follows that

$$\omega_1 = \omega_1' = \omega_2 = \omega \text{ (say)}.$$

That is the incident, reflected and refracted waves all have the same frequency.

Since equation (5) holds for all points of the interface ($z = 0$), we msut have

$$(\mathbf{k}_1 \cdot \mathbf{r})_{z=0} = (\mathbf{k}_1' \cdot \mathbf{r})_{z=0} = (\mathbf{k}_2 \cdot \mathbf{r})_{z=0}\,. \qquad ...(7)$$

This equation is independent of the nature of the boundary conditions and contain the kinematic aspects of reflection and refraction.

Writting equation (7) is somewhat expanded from *i.e.*

$$k_{1x}x + k_{1y}y = k'_{1x}x + k'_{1y}y = k_{2x}x + k_{2y}y \qquad ...(8)$$

we get

$$\left.\begin{aligned} k_{1x} &= k'_{1x} = k_{2x} \quad ...(a) \\ k_{1y} &= k'_{1y} = k_{2y} \quad ...(b) \end{aligned}\right\} \qquad ...(9)$$

Since incident beam is in *XZ* plane $k_{1y} = 0$, therefore equation (9) implies $k'_{1y} = k_{2y} = 0$, that is both $\mathbf{k}_1'$ and $\mathbf{k}_2$ also lie in *XZ* plane. As normal **n** is along *Z* axis, thus we conclude that all the ***three wave vectors and normal to the interface* n *all lie in the same plane.*** In other words ***the incident, reflected; refracted waves and the normal to the interface all lie in the same plane.***

Furthermore fig. 9.3, we get

$$\left.\begin{aligned} \mathbf{k}_1 \cdot \mathbf{r} &= k_1 (x \sin\theta_i + z\cos\theta_i) \quad ...(a) \\ \mathbf{k}_1' \cdot \mathbf{r} &= k_1' (x \sin\theta_r' - z\cos\theta_r') \quad ...(b) \\ \mathbf{k}_2 \cdot \mathbf{r} &= k_1 (x \sin\theta_r - z\cos\theta_r) \quad ...(c) \end{aligned}\right\} \qquad ...(10)$$

Substitution values from (10a) and (10b) in equation (7) viz.

$$(\mathbf{k}_1 \cdot \mathbf{r})_{z=0} = (\mathbf{k}_1' \cdot \mathbf{r})_{z=0}$$

we get

$$k_1 x \sin\theta_i = k_1' x \sin\theta_r'$$

or

$$k_1 \sin\theta_i = k_1' \sin\theta_r' \qquad ...(11)$$

Since wave vector $\mathbf{k}_1$ and $\mathbf{k}_1'$ lie in the same medium, hence $k_1 = k_1' = \omega\sqrt{(\mu_1\varepsilon_1)} = (\omega/v_1)\, v_1$, being phase velocity of electromagnetic wave in medium 1. Therefore equation (11) gives

$$\sin\theta_i = \sin\theta_r' \;\text{ or }\; \theta_i = \theta_r' \qquad ...(12)$$

i.e. the angle of incident (θ_i), *is equal to the angle of reflection* (∂_r'),

Now substituting values of $\mathbf{k}_1 \cdot \mathbf{r}$ and $\mathbf{k}_2 \cdot \mathbf{r}$ from equation (10) in (7) viz

$$(\mathbf{k}_1 \cdot \mathbf{r})_{z=0} = (\mathbf{k}_2 \cdot \mathbf{r})_{z=0}$$

we get

$$k_1 x \sin\theta_i = k\, x \sin\theta_r$$

i.e.

$$k_1 \sin \theta_i = k_2 \sin \theta_r$$

or

$$\frac{\sin \theta_i}{\sin \theta_r} = \frac{k_2}{k_1} = \frac{\omega_2 \sqrt{(\mu_2 \varepsilon_2)}}{\omega_1 \sqrt{(\mu_1 \varepsilon_1)}} = \frac{\sqrt{(\mu_2 \varepsilon_2)}}{\sqrt{(\mu_1 \varepsilon_1)}} \quad (\text{since } \omega_1 = \omega_2 = \omega)$$

This gives

$$\frac{\sin \theta_i}{\sin \theta_r} = \frac{n_2}{n_1}$$

(since refractive index, $n = \sqrt{(\mu\, \varepsilon)} = (c/v)$)

This is well known *Snell's law of refraction.*

9.13. Rectangular Wave Guide

The most commonly used wave guide is that of rectangular cross-section having inner dimension a and b as shown in fig. 9.21.

The solution of two dimensional wave equation

$$(\nabla_{\perp}^2 \neq k_c^2)\,\psi = 0 \qquad ...(1)$$

can be carried out in rectangular coordinates as follows :

***TE* Mode.** For *TE* mode $E_z = 0$: hence equation (1) is to be written for B_z ; which takes the form

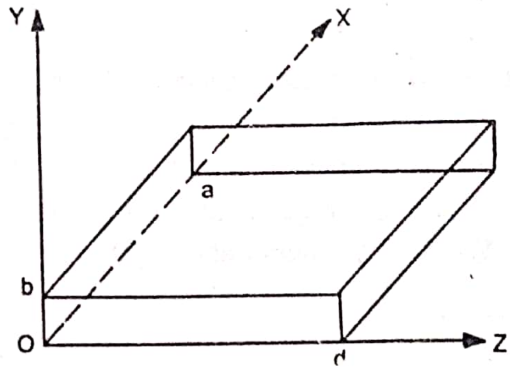

Fig. 9.12

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) B_z = 0 \qquad ...(2)$$

The boundary conditions are

$$\left.\frac{\partial B_z}{\partial n}\right|_S = 0$$

i.e.

$$\frac{\partial B_z}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = a$$

and

$$\frac{\partial B_z}{\partial y} = 0 \text{ at } y = 0 \text{ and } y = b$$

We shall solve equation (2) by the method of separation of variables.

Therefore writing

$$B_z(x, y) = X(x)\,Y(y) = XY \qquad ...(3)$$

where X is a function of x only and Y is a function of y only.

Susbtituting equation (3) in (2) and dividing by XY, we get

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + k_c^2 = 0$$

or

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + k_c^2 = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}$$

In above equation L.H.S. is a function of x only, while R.H.S. is a function of y only. Hence this equation will be satisfied if both sides are equal to a constant say p^2 *i.e.*

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + k_c^2 = +p^2$$

or

$$\frac{\partial^2 X}{\partial x^2} + (k_c^2 - p^2)\,X = 0$$

or

$$\frac{\partial^2 X}{\partial x^2} + q^2 X = 0 \qquad ...(4)$$

where

$$q^2 = k_c^2 - p^2 \qquad ...(5)$$

and

$$-\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = p^2$$

or

$$\frac{\partial^2 Y}{\partial y^2} + p^2 Y = 0 \qquad ...(6)$$

The solution of equation (4) and (6) are

$$X(x) = A\cos qx + B\sin qx \qquad ...(7)$$

$$Y(y) = C\cos py + D\sin py \qquad ...(8)$$

where A, B, C, D are arbitrary constants.

We have the boundary conditions

$$\frac{\partial B_z}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = a$$

$$\frac{\partial B_z}{\partial y} = 0 \text{ at } y = 0 \text{ and } y = b$$

These conditions are equivalent to

$$\frac{\partial X}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = a$$

and

$$\frac{\partial Y}{\partial y} = 0 \text{ at } y = 0 \text{ and } y = b.$$

Differentiating equations (7) and (8), we get

$$\frac{\partial X}{\partial x} = -Aq\sin qx + Bq\cos qx$$

$$\frac{\partial Y}{\partial y} = -Cp\sin qy + Dp\cos py$$

Applying boundary condition $\left.\frac{\partial Z}{\partial x}\right|_{x=a} = 0$, we get

$Bq = 0$ ∴ This gives $B = 0$

Now applying boundary condition $\left.\frac{\partial X}{\partial x}\right|_{x=a} = 0$, we get

$$-Aq\sin qa = 0$$

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We must take $A \neq 0$ since otherwise $X = 0$ and $B_z = 0$. Hence

$$\sin qa = 0 \text{ or } qa = m\pi, \text{ that is,}$$

$$q = \frac{m\pi}{a} \quad (m \text{ integer}) \qquad \text{...(11)}$$

In precisely the same manner we conclude that $D = 0$ and p must be restricted to values $p = n\pi/b$ where n is an integer. In this way be obtain the solutions

$$X(x) = A\cos\left(\frac{m\pi}{a}\right)x; \quad Y(y) = C\cos\left(\frac{n\pi}{b}\right)y \qquad \text{...(12a)}$$

where $\qquad m = 1, 2, 3, \ldots.. \quad n = 1, 2, 3, \ldots.$

and $\qquad k_c^2 = p^2 + q^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$

The solution for $B_z(x, y)$ is consequently

$$B_z(x, y) = B_0 \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \qquad \text{...(13a)}$$

when $\qquad (k_c^2)_{mn} = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{1/2} \qquad \text{...(13b)}$

Here the indices mn specify the mode. The *cut off frequency* ω_{mn} is given by

$$\omega_{mn} = \pi c\left[\frac{m^2}{a^2} + \frac{n^2}{b^2}\right]^{1/2} \qquad \text{...(14)}$$

The modes corresponding to m and n are represented as TE_{mn}. The case $m = n = 0$ gives a static field which does not represent a wave propagation ; hence the mode TE_{00} represents non-trivial solution. If $a > b$, the lowest *cut off frequency* results for $m = 1$ and $n = 0$,

i.e. $\qquad \omega_{10} = \frac{\pi c}{a} \text{ or } (k_c)_{10} = \frac{\pi}{a} \qquad \text{...(15)}$

The mode (TE_{10}) represents the *dominant TE mode* and is the one used in most practical situations. The values E_x, E_y, B_x and B_y for *TE* mode may be obtained from equations (22)–(25) of preceding section by substituting the solution for B_z, which is

$$B_z(r, t) = B_z(x, y)\, e^{ik_g z - i\omega t}$$

$$= B_0 \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b}\, e^{ik_g z - i\omega t} \qquad \text{...(16)}$$

Thus we have

$$\left.\begin{aligned}
E_x &= \frac{in\pi\omega}{k_c^2 b} B_0 \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{ik_g z - i\omega t}\\
E_y &= \frac{im\pi\omega}{k_c^2 a} B_0 \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{ik_g z - i\omega t}\\
B_x &= -\frac{im\pi k_g}{k_c^2 a} B_0 \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{ik_g z - i\omega t}\\
B_y &= -\frac{in\pi k_g}{k_c^2 a} B_0 \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{ik_g z - i\omega t}
\end{aligned}\right\} \quad ...(17)$$

For *TE* mode, these equations yield

$$\left(\text{put } m = 1,\ n = 0,\ k_c^2 = \frac{\pi^2}{a^2}\right)$$

$$\left.\begin{aligned}
&E_x = B_y = E_z = 0;\ B_z = B_0 \cos\left(\frac{\pi x}{a}\right) e^{ik_g z - i\omega t}\\
\text{and} \quad &B_x = -\frac{ik_g a}{a} B_0 \sin\left(\frac{\pi x}{a}\right) e^{ik_g z - i\omega t}\\
&E_y = \frac{i\omega a}{\pi c} B_0 \sin\left(\frac{\pi x}{a}\right) e^{ik_g z - i\omega t}
\end{aligned}\right\} \quad ...(18)$$

The presence of a factor i in B_x (and E_y) means that there is a spatial (or temporal) phase difference of $\pi/2$ between B_x (and E_y) and B_z in the propagation.

***TM* Mode :** For *TM* mode $B_z = 0$; hence equation (1) is to be written for E_z which takes the form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) E_z = 0 \quad ...(19)$$

The boundary conditions are $E_z = 0$ at $x = 0$, $x = a$, $y = 0$ and $y = b$.

Solving equation (19) as for *TE* case, we note that the solution of equation (19) is of the form

$$E_z(x, y) = E_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad ...(20)$$

where k_c^2 and hence ω_{mn} are still given by equations (12b) and (14). This implies that *TE and TM modes of a rectangular guide have the same set of cut off frequencies.* However in this case $m = 1$ and $n = 0$ represents non-trival solution since this gives $E_z = 0$ and hence all components of **E** and **B** will be zero.

It is obvious that in this case the lowest mode has $m = n = 1$ and may be represented by TM_{11}. The cut off frequency of lowest mode is given by

$$\omega_{11} = \pi c\left[\frac{1}{a^2} + \frac{1}{b^2}\right]^{1/2} = \frac{\pi c}{a}\left[1 + \frac{a^2}{b^2}\right]^{1/2} \quad ...(21)$$

Since $a < b$, therefore the *cut off frequency of lowest TM mode is greater than that of the lowest TE mode by the factor* $\left[1 + \left(\frac{a^2}{b^2}\right)\right]^{1/2}$ The fields E_x, E_y, B_x, B_y for *TM* mode may be obtained from equations (17)–(22) of preceding section if we substitute

$$E_z(\mathbf{r}, t) = E_0(x, y)\, e^{ik_g z - i\omega t}$$

or

$$E_z(\mathbf{r}, t) = E_0 \sin\left[\frac{m\pi x}{a}\right] \sin\left[\frac{n\pi y}{b}\right] e^{-ik_g z - i\omega t} \quad ...(22)$$

Thus we have

$$\left.\begin{aligned} E_x &= \frac{im\pi k_g}{k_c^2 a} E_0 \cos\left[\frac{m\pi x}{a}\right] \sin\left[\frac{n\pi y}{b}\right] e^{ik_g z - i\omega t} \\ E_y &= \frac{in\pi k_g}{k_c^2 b} E_0 \sin\left[\frac{m\pi x}{a}\right] \cos\left[\frac{n\pi y}{b}\right] e^{ik_g z - i\omega t} \\ B_x &= -\frac{i\omega n\pi}{k_c^2 bc} E_0 \sin\left[\frac{m\pi x}{a}\right] \cos\left[\frac{n\pi y}{b}\right] e^{ik_g z - i\omega t} \\ B_y &= \frac{i\omega n\pi}{k_c^2 ac^2} E_0 \cos\left[\frac{m\pi x}{a}\right] \sin\left[\frac{n\pi y}{b}\right] e^{ik_g z - i\omega t} \end{aligned}\right| \quad ...(22)$$