6

Elasticity

6.1 Elasticity

Elasticity is the property by virtue of which material bodies regain their original shape and size after the external deforming forces are removed. When an external force acts on a body, there is change in its length, shape and volume. The body is said to be strained.

When this external force is removed, the body regains its original shape and size. Such bodies are called elastic bodies. Steel, glass ivory, quartz etc. are elastic bodies. The bodies which do not regain their original shape and size are called plastic bodies. No body is either completely elastic or completely plastic. The property of elasticity is different in different substances. Steel is more elastic than rubber. Liquids and gases are highly elastic.

6.2 Definitions

Elasticity is defined as the property by which a body regains its original position when the forces are withdrawn.

The opposite of elasticity is plasticity. No substance is perfectly elastic or perfectly plastic.

Stress : When a force is applied on a body, there will be relative displacement of the particles and due to the property of elasticity the particles tend to regain their original position. Stress is defined as the restoring force per unit area.

Normal Stress. Restoring force per unit area prependicular to the surface is called normal stress.

Tangential Stress. Restoring force parallel to the surface per unit area is called tangential stress.

Strain. The ratio of the change in shape to the original shape is called strain. There are three types of strain :

Longitudinal Strain. The ratio of change in length to original length is called longitudinal strain $\left(\frac{l}{L}\right)$

Shearing Strain. Shearing strain is defined as the angle of shear measured in radians.

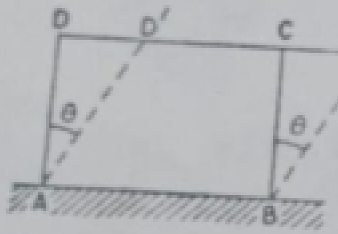


Fig. 6.1

The surface AB is fixed and a force is applied parallel to the surface CD, so that the body is sheared by an angle θ. The angle θ measured in radians is called the shearing strain (θ) (Fig. 6.1).

Volume Strain. The ratio of the change in volume to original volume is called volume strain $\left(\frac{v}{V}\right)$.

Elastic limit. The maximum stress up to which a body exhibits the property of elasticity is called elastic limit. If the applied force exceeds the maximum stress limit, the body does not regain its original position completely after the external forces are withdrawn.

Hooke's law. It states that within the elastic limit stress is directly proportional to straln.

$$\text{Stress} \propto \text{Strain}$$

$$\text{Stress} = E \times \text{Strain}$$

$$\therefore \quad \frac{\text{Stress}}{\text{Strain}} = E$$

E is a constant called the modulus of elasticity.

(*i*) **Young's modulus of elasticity** (Y). It is defined as the ratio of normal stress to longitundinal strain.

$$Y = \frac{\text{Normal Stress}}{\text{Longitudinal Strain}} = \frac{\frac{F}{\text{area}}}{\frac{l}{L}} = \frac{FL}{al}$$

(*ii*) **Modulus of rigidity** (η). It is defined as the ratio of tangential stress to shearidg strain.

$$\eta = \frac{\text{Tangential Stress}}{\text{Shearing Stain}} = \frac{\frac{F}{\text{area}}}{\theta} = \frac{F}{a \times \theta}$$

(*iii*) **Bulk modulus of elasticity** (K). It is denned as the ratio of normal stress to volume strain,

$$K = \frac{\text{Normal Stress}}{\text{Volume Strain}} = \frac{\frac{F}{\text{area}}}{\frac{v}{V}} = \frac{FV}{av} = \frac{pV}{v}$$

$$= \frac{9.8 \times 0.5}{2 \times 10^{-6}} \left(\frac{0.6}{15 \times 10^{-3}} - \frac{0.04}{5 \times 10^{-3}} \right)$$

$$= 2.45 \times 32 \times 10^{6} \text{ N/m}^2$$

$$Y = \frac{\text{Change in stress}}{\text{Change in strain}} = \frac{2.45 \times 32 \times 10^{6}}{4 \times 10^{-4}}$$

$$= 19.6 \times 10^{10} \text{ N/m}^2$$

6·6 Poisson's Ratio

Whenever a body is subjected to a force in a particular direction, there is change in dimensions of the body in the other two perpendicular directions. This is called lateral strain. Lateral strain is proportional to the size of the body in the other two perpendicular directions.

Let α be the longitudinal strain per unit stress and β the lateral strain per unit stress, Within the elastic limit.

$$\beta \propto \alpha$$

or

$$\beta = \sigma \alpha$$

$\therefore$ Poission's ratio

$$\sigma = \frac{\beta}{\alpha}$$

Therefore, Poisson's ratio* is **defined as the ratio of lateral strain per unit stress to the longitudinal strain per unit stress.**

Consider a wire of length L and diameter D. The wire is fixed at one end and a force is applied at the other end. Consequently the length of the wire increases and the diameter of the wire decreases (Fig, 6·4).

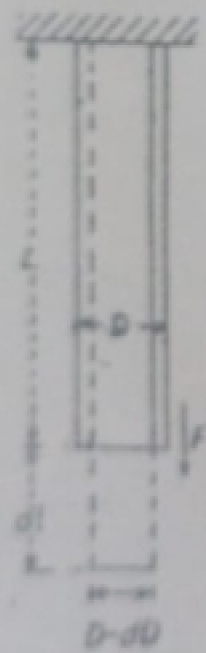


Fig. 6·4

Suppose, increase in length is dL, decrease in diameter is dD.

$$\therefore \quad \sigma = \frac{\beta}{\alpha}$$

$$\sigma = -\frac{dD/D}{dL/L}$$

$$\sigma = -\left(\frac{dD}{dL}\right)\left(\frac{L}{D}\right) \quad \ldots(i)$$

The value of **Poisson's ratio mainly depends upon the nature of the material of the body.** σ has no units as it is a ratio of two numbers. For most of the substances, the value of σ varies between 0·2 and 0·4.

If the volume of the wire remains unchanged after the force has been applied, then

Initial volume of the wire,

$$V = \left(\frac{\pi D^2}{4}\right) L$$

Differentiating equation (*ii*)

$$dV = \frac{\pi}{4}\left[D^2\, dL + 2LD\, dD\right]$$

If $\quad dV = 0$

$$D^2\, dL + 2LD\, dD = 0$$

$$\frac{dD}{dL} \times \frac{L}{D} = -\frac{1}{2}$$

But $\quad \sigma = -\left(\frac{dD}{dL} \times \frac{L}{D}\right)$

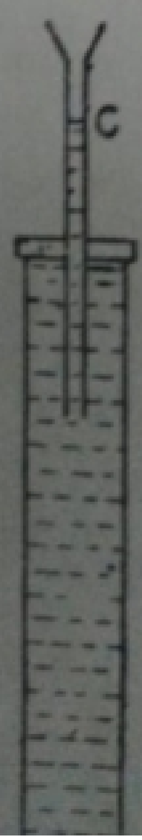
$\therefore \quad \sigma = -\left(-\frac{1}{2}\right)$

$$\sigma = \frac{1}{2}$$

This is the maximum possible value of Poisson's ratio.

6·7. Poisson's Ratio of Rubber

A long rubber tube of length about 1 metre is suspended vertically and a weight hanger is fixed at its lower end (Fig. 6·5). The rubber tube is completely filled with water so that the water meniscus is visible in the graduated glass tube *C*.



When a weight is placed in the weight hanger, the length of the tube increases and its area of cross section decreases. The internal volume of the tube also increases. Consequently the level of water in the tube *C* falls. The distance through which the pointer *P* moves is measured with the help of a travelling microscope. Suppose, increase in the length of the tube $= dL$

Initial volume $= V$

Initial length $= L$

Initial area of cross section $= A$

Bulk Modulus. (Relation between E, Y and σ)

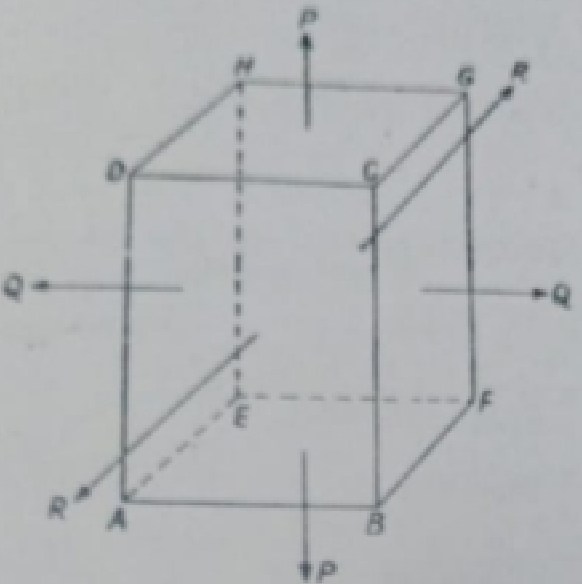


Fig. 6·6.

Consider a block of rectangular cross section having length l, breadth b and thickness t. Let the block be subjected to outward stress as shown in Fig. 6·6. Here, P is the stress acting on the faces $ABFE$ and $CDHG$. Here Q is the stress acting on the faces $AEHD$ and $BCGF$ and R is the stress acting on the faces $ABCD$ and $EFGH$.

Each stress produces an extension in its own direction and a lateral contraction in the other two perpendicular directions. Let α be the longitudinal strain per unit stress and β the lateral strain per unit stress. Poisson's ratio $\sigma = \dfrac{\beta}{\alpha}$

Increase in length due to the stress P is $P\alpha l$ and corresponding contraction in breadth and thickness will be $P\beta b$ and $P\beta t$ respectively.

Similarly, due to the stress Q, increase in breadth is $Q\alpha b$ and corresponding decrease in length and thickness will be $Q\beta l$ and $Q\beta t$ respectively. Also due to the stress R, increase in thickness is $R\alpha t$ and decrease in length and breadth will be $R\beta l$ and $R\beta b$.

Final length $= l+P\alpha l-Q\beta l-R\beta l$

Final breadth $= b+Q\alpha b-P\beta b-R\beta b$

Final thickness $= t+R\alpha t-P\beta t-Q\beta t$

Final volume $= lbt[1+P\alpha-(Q+R)\beta]$

$\times[1+Q\alpha-(P+R)\beta]\times[1+R\alpha-(P+Q)\beta$

$= lbt[1+(P+Q+R)\alpha-2\beta(P+Q+R)]$

$= lbt[1+(P+Q+R)(\alpha-2\beta)]$

Change in volume $= lbt(P+Q+R)(\alpha-2\beta)$

If the stresses are equal, $P = Q = R$

Change in volume $= lbt(3P)(\alpha-2\beta)$

Strain $= \dfrac{lbt(3P)(\alpha-2\beta)}{lbt} = 3P(\alpha-2\beta)$

Bulk modulus, $K = \dfrac{P}{3P(\alpha-2\beta)}$

or $$K = \frac{1}{3(\alpha-2\beta)} \quad \ldots(i)$$

or $$K = \frac{1}{3\alpha\left(1-\frac{2\beta}{\alpha}\right)}$$

$$K = \frac{1}{3\alpha(1-2\sigma)} \quad \ldots(ii)$$

But $\dfrac{1}{\alpha} = Y$

$\therefore$ $$K = \frac{Y}{3(1-2\sigma)} \quad \ldots(iii)$$

6·10 Modulus of Rigidity

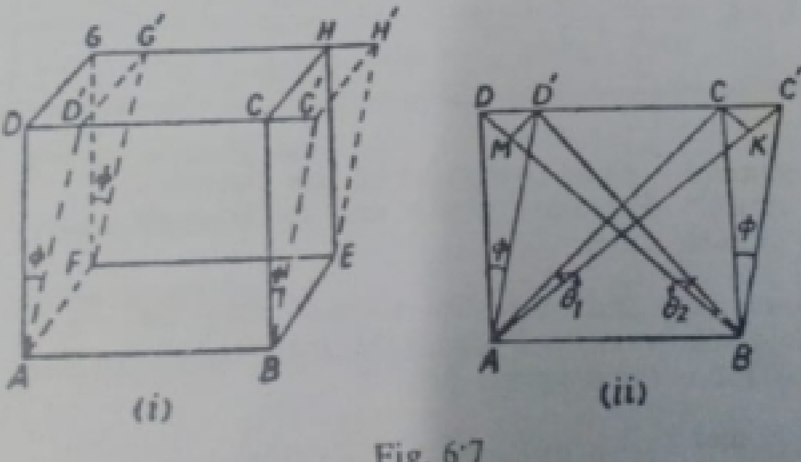


Fig. 6·7

Consider a cubical block *ABCDEFGH* of length *l*. The f *ABEF* is kept fixed and a tangential stress *T* is applied on the *CDGH*. The block gets sheared through an angle φ [Fig. 6·7 All the particles in the plane *CDGH* remain in the same plane displaced equally in the same direction. Consequently, *D* is sh to *D′*, *C* to *C′*, *G* to *G′*, and *H* to *H′*.

The diagonal *AC* gets extended to *AC′* while the diagonal gets compressed to *BD′*. Extension strain is θ and also compre strain is θ. [Fig. 6·7 (*ii*)]

∴ [illegible]

$$\frac{l}{\sqrt{2}} = (L\sqrt{2})\,(\alpha+\beta)$$

∴ $$\frac{l}{L} = 2T\,(\alpha+\beta)$$

But $$\phi = \frac{l}{L}$$

∴ $$\phi = 2T(\alpha+\beta)$$

or $$\frac{T}{\phi} = \frac{1}{2(\alpha+\beta)}$$

∴ $$\eta = \frac{T}{\phi} = \frac{1}{2(\alpha+\beta)}$$

Hence $$\eta = \frac{1}{2(\alpha+\beta)} \qquad ...(i)$$

$$\eta = \frac{1}{2\alpha(1+\sigma)}$$

But $$\frac{1}{\alpha} = Y$$

∴ $$\eta = \frac{Y}{2(1+\sigma)} \qquad ...(ii)$$

Relation between Elastic Constants (Y, η, K and σ)

The various equations are

$$Y = \frac{1}{\alpha} \qquad ...(i)$$

$$K = \frac{1}{3(\alpha-2\beta)} \qquad ...(ii)$$

$$\eta = \frac{1}{2(\alpha+\beta)} \qquad ...(iii)$$

From equations (*ii*) and (*iii*)

$$\alpha-2\beta = \frac{1}{3K} \qquad ...(iv)$$

$$2\alpha+2\beta = \frac{1}{\eta} \qquad ...(v)$$

Adding (*iv*) and (*v*) and simplifying

$$3\alpha = \frac{1}{3K} + \frac{1}{\eta}$$

$$\frac{3}{Y} = \frac{1}{3K} + \frac{1}{\eta} \qquad ...(vi)$$

From equation (*vi*)

$$Y = \frac{9\eta K}{\eta + 3K} \qquad \text{...(vii)}$$

$$\eta = \frac{3KY}{9K - Y} \qquad \text{...(viii)}$$

$$K = \frac{Y\eta}{9\eta - 3Y} \qquad \text{...(ix)}$$

From equations (*vii*), (*viii*) and (*ix*), values of Y, η and K can be calculated if any two values are known.

From equation (*iii*)

$$\eta = \frac{1}{2(\alpha + \beta)} = \frac{1}{2\alpha\left(1 + \frac{\beta}{\alpha}\right)}$$

$$\eta = \frac{Y}{2(1+\sigma)} \qquad \text{...(x)}$$

or

$$\sigma = \left(\frac{Y}{2\eta}\right) - 1 \qquad \text{...(xi)}$$

Multiplying equation (*iv*) by 2 and subtracting it from equation (*v*) we get

$$6\beta = \frac{1}{\eta} - \frac{2}{3K}$$

or

$$\beta = \frac{3K - 2\eta}{18\eta K} \qquad \text{...(xii)}$$

The relation for Poisson's ratio can be found in terms of K and η.

$$\sigma = \frac{\beta}{\alpha} = \beta Y = \left(\frac{3K - 2\eta}{18\eta K}\right)\left(\frac{9\eta K}{\eta + 3K}\right)$$

$$\sigma = \frac{3K - 2\eta}{2(\eta + 3K)} \qquad \text{...(xiii)}$$

6·12 Alternative Method

The relation between Y, η and K can also be obtained with the help of the following table.

Consider a unit cube, which is subjected to outward elongational force P on each face. Let σ be the Poisson's ratio for the material. In the table, the values of applied stress and the corresponding strains produced along the three perpendicular axes are shown. For a stress P the longitudinal strain produced $= \frac{P}{Y}$ in its own direction and the corresponding strains in the other two

$$\eta = \frac{4\pi M R^2 l}{r^4 t^2},$$

Here

$$M = 1{\cdot}2 \text{ kg}$$
$$R = 0{\cdot}05 \text{ m}$$
$$l = 1 \text{ m}$$
$$r = 0{\cdot}72 \times 10^{-3} \text{ m}$$
$$t = 1{\cdot}98 \text{ s}$$

$$\therefore \quad \eta = \frac{4\pi \times 1{\cdot}2 \times (0{\cdot}05)^2 \times 1}{(0{\cdot}72 \times 10^{-3})^4 \times (1{\cdot}98)^2}$$

$$\eta = 3{\cdot}579 \times 10^{10} \text{ N/m}^2.$$

6·18 Bending of Beams

A beam is defined as a structure of uniform cross section, whose length is large in comparison to its breadth and thickness. For such a structure, the shearing stress for any given cross section is negligible. Beams are used in the construction of bridges or for purposes of supporting heavy loads. They are commonly used in the structure of multistoried buildings. A beam can be used in a horizontal position or as columns and pillars in a vertical position.

Consider a beam of uniform rectangular cross section. Let the beam be subjected to deforming forces so that it bends (Fig. 6·13).

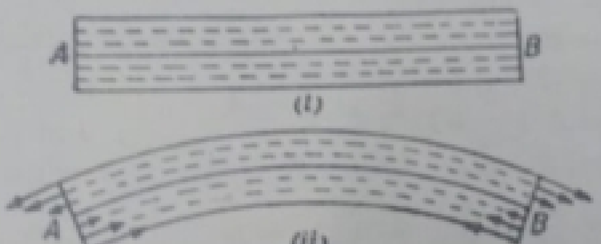


Fig. 6·13

In the initial position of the beam, the various filaments constituting the beam are in parallel layers of equal length. When the deforming forces are applied, the beam bends as shown in Fig. 6·13 (*ii*). Above the layer *AB*, the filaments are elongated while below *AB* they are compressed. The length of the layer *AB* remains unaltered. It is called the **neutral axis**. The surface containing the neutral axis and perpendicular to the plane of bending is called the neutral surface. Further the change in length of any filament (extension or contraction) is proportional to the distance of the filament from the neutral axis.

6·19 Bending Moment

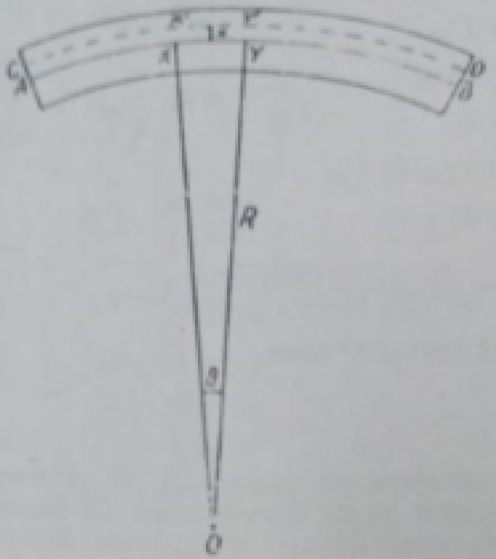


Fig. 6·14

Consider a beam under the action of deforming forces. Du to the property of elasticity of the material, a restoring coup acts on the beam. In the equilibrium position, the bending coup is equal and opposite to the restoring couple. *The moment of th restoring couple is called the bending moment.*

Consider a filament $X'Y'$ at a distance x from the neutral axi Here $X'Y'$ has been extended. In Fig. 6·14,

$$XY = R\theta$$

and

$$X'Y' = (R+x)\,\theta$$

$\therefore$ Increase in length $= X'Y' - XY$

$$= (R+x)\,\theta - R\theta = x\theta$$

Strain $= \dfrac{x\theta}{R\theta} = \dfrac{x}{R}$

Young's modulus $Y = \dfrac{\text{Stress}}{\text{Strain}}$

$$\text{Stress} = Y \times \text{Strain} = \frac{Yx}{R}$$

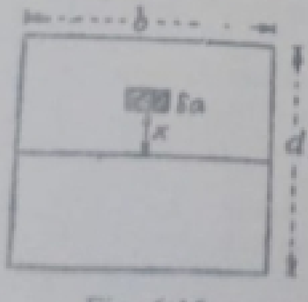


Fig. 6·15

The cross section of the beam is show in Fig. 6·15.

The force acting on the element o area of cross section δa

$$= \text{stress} \times \delta a$$

$$F = \frac{Yx}{R} \times \delta a$$

The forces producing elongation act on the upper half outward and those producing contraction act on the lower half inward. These two forces constitute a couple.

Moment of the force about the neutral axis.

$$=\left[\frac{Yx}{R}\times\delta a\right].x=\left(\frac{Yx^2}{R}\right)\delta a$$

The moment of all the forces about the neutral axis

$$=\sum\frac{Yx^2}{R}\delta a$$

$$=\frac{Y}{R}\sum x^2\delta a$$

Here $\Sigma x^2\delta a=aK^2=I_g$

where a is the area of cross section of the beam and K is the radius of gyration and $aK^2=I_g$. I_g is called the geometrical moment of inertia of the beam. Hence, Bending moment $=\dfrac{YI_g}{R}$

Special Cases

1. **Rectangular cross section**

If the breadth of the beam is b and thickness is d, then

$$a=b\times d$$

and

$$K^2=\frac{d^2}{12}$$

$$\therefore\quad I_g=aK^2=\frac{(bd)d^2}{12}=\frac{bd^3}{12}$$

2. **Circular cross section**

Here

$$a=\pi r^2 \text{ and } K^2=\frac{r^2}{4}$$

$$\therefore\quad I_g=aK^2=\frac{\pi r^2\times r^2}{4}=\frac{\pi r^4}{4}$$

Flexural rigidity. It is defined as the external bending moment required to produce a unit radius of curvature. Therefore, flexural Rigidity $=YI_g$.

6·20 Basic Assumptions for Theory of Bending

1. The length of the beam is large in comparison to the thickness of the beam. Consequently, the shearing stresses are negligible.

2. The stress is proportional to strain. For elongation as well as compression, the Young's modulus of the beam has the same value.

6·22 Beam supported at its Ends and Loaded in the Middle

Non-uniform bending

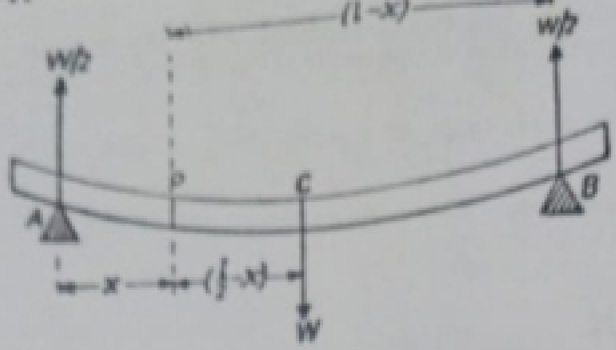


Fig. 6.17

Consider a beam (of negligible mass) supported on the two knife edges A and B (Fig. 6.17). The length of the beam between the two points A and B is l. It is loaded in the middle with a load W. The reaction at each support is $W/2$ in the vertically upward direction. Consider a cross section of the beam at P, at a distance x from A. For equilibrium

$$\frac{YI_g}{R} = W\left[\frac{l}{2}-x\right]-\frac{W}{2}(l-x)$$

$$\frac{YI_g}{R} = -\frac{W}{2}\times x$$

$$\therefore \qquad \frac{1}{R} = -\frac{Wx}{2YI_g} \qquad \text{...(i)}$$

But

$$\frac{1}{R} = \frac{d^2y}{dx^2}$$

$$\therefore \qquad \frac{d^2y}{dx^2} = -\frac{Wx}{2YI_g} \qquad \text{...(ii)}$$

Integrating with respect to x

$$\frac{dy}{dx} = -\frac{Wx^2}{4YI_g}+C_1 \qquad \text{...(iii)}$$

At $x=\frac{l}{2}$, $\frac{dy}{dx}=0$

$$0 = -\frac{Wl^2}{16YI_g}+C_1$$

$$C_1 = \frac{Wl^2}{16YI_g} \qquad \text{...(iv)}$$

Substituting this value in equation (*iii*)

$$\frac{dy}{dx} = -\frac{Wx^2}{14YI_g}+\frac{Wl^2}{16YI_g}$$

integrating with respect to x

$$y = -\frac{Wx^3}{12YI_g} + \frac{Wl^2x}{16YI_g} + C_3 \quad \text{...(v)}$$

At $\quad x = 0,\ y = 0$

$\therefore \quad C_3 = 0$

$$\therefore \quad y = \frac{-Wx^3}{12YI_g} + \frac{Wl^2x}{16YI_g} \quad \text{...(vi)}$$

At the mid point C,

$$x = \frac{l}{2} \text{ and}$$

$$y = \frac{-Wl^3}{96YI_g} + \frac{Wl^3}{32\,YI_g}$$

$$y = \frac{Wl^3}{48YI_g} \quad \text{...(vii)}$$

(1) For a beam of rectangular cross-section

$$I_g = \frac{bd^3}{12}$$

$$\therefore \quad y = \frac{Wl^3}{4Ybd^3} \quad \text{...(viii)}$$

(2) For a beam of circular cross section

$$I_g = \frac{\pi r^4}{4}$$

$$\therefore \quad y = \frac{Wl^3}{12\pi Yr^4} \quad \text{...(ix)}$$

6·26 Depression of Beam under its Own Weight

Let a heavy beam of mass w per unit length be supported at the points A and B. The weight of the beam between the two knife edges, $W_1 = wl$.

The reaction at each support $= \frac{wl}{2}$ in the vertically upward direction. The beam bends under its own weight (Fig. 6·18).

Consider a cross-section of the beam at P at a distance x from A. The weight of the portion PB of the beam is $w\,(l-x)$. This weight acts at $w\left(\frac{l-x}{2}\right)$ from P. The reaction $\frac{wl}{2}$ at B produces a deflec... moment.

CHAPTER 2

Viscosity

2.1 INTRODUCTION

When two parallel layers of a liquid are moving with different velocities, they experience tangential forces which tend to retard the faster layer and accelerate the slower layer. These forces are called forces of viscosity. Consider two layers of liquid separated by a distance dz (Fig. 2.1). Let v and $v + dv$ be the velocities of two layers. So the velocity gradient is dv/dz. Let A be the surface area of the layer. The viscous force is directly proportional to the surface area A and velocity gradient dv/dz

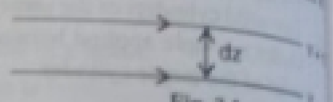


Fig. 2.1

i.e., $F \propto A \dfrac{dv}{dz}$ or $F = \eta A \dfrac{dv}{dz}$

where η is a constant for the liquid and called coefficient of viscosity. If $A = 1$ and $dv/dz = 1$, we have $F = \eta$.

The coefficient of viscosity is defined as the tangential force per unit area required to maintain a unit velocity gradient.

Unit of η is Nsm^{-2}. It is called the pascal second.

$$\text{Dimensions of } [\eta] = \frac{[F]}{[A][(dv/dz)]} = \frac{MLT^{-2}}{L^2\,(LT^{-1}/L)} = ML^{-1}T^{-1}$$

2.2 STREAMLINE FLOW AND TURBULENT FLOW

Consider a liquid flowing in a pipe. Let the velocity of flow be v_1 at A, v_2 at B and v_3 at C (Fig. 2.2). If as time passes, the velocities at A, B, and C are constant in magnitude and direction, then the flow is said to be steady. In a steady flow, each particle follows exactly the same path and has exactly the same velocity as its predecessor. In such a case, the liquid is said to have an orderly or *streamline flow*.

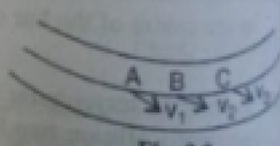


Fig. 2.2

The line ABC is called a streamline, which is the path followed by an orderly procession of particles. The tangent to the streamline at any point gives the velocity of the liquid at that point.

The flow is steady or streamlined only as long as the velocity of the liquid does not exceed a limiting value, called the *critical velocity*. When the external pressure causing the flow of the liquid is excessive, the motion of the liquid takes place with a velocity greater than the critical velocity and the motion becomes unsteady or *turbulent*. This causes eddies and whirlpools in the motion of the liquid. This turbulent motion is also known as *vortex motion*.

The distinction between streamline flow and turbulent flow can be demonstrated by injecting a jet of ink axially in a wider tube in which water is made to flow axially. When the velocity of the liquid is small, the ink will move in a straight line. As the speed of flow is incre... critical velocity, the ink will spread out, showing that the motion has become turbulent ... the

Definition of critical velocity. *Critical velocity of a liquid is the velocity below which the motion of the liquid is orderly and above which the motion of the liquid becomes turbulent.*

Expression for the critical velocity. *The critical velocity of a liquid may depend upon* (*i*) the coefficient of viscosity of the liquid (η), (*ii*) the density of the liquid (ρ) and (*iii*) the radius *r* of the tube through which the liquid is flowing. We may write

$$v_c = k\,\eta^a \rho^b r^c$$

where *k* is constant called *Reynold's number.*

Writing the dimensions of these quantities,

$$[LT^{-1}] = [ML^{-1}T^{-1}]^a [ML^{-3}]^b [L]^c$$

$$[LT^{-1}] = [M^{a+b} L^{-a-3b+c} T^{-a}]$$

$$\therefore \quad a + b = 0 \,;\, -a - 3b + c = 1 \text{ and } -a = -1$$

From these equations we have, $a = 1$, $b = -1$ and $c = -1$.

$$\therefore \quad v_c = \frac{k\,.\,\eta}{\rho\, r}$$

Significance of Reynold's number :

$$k = \frac{v_c \rho\, r}{\eta}$$

The significance of the Reynold's number *k* is that its value determines the nature of flow ... liquid through a tube. In the case of apparatus, geometrically similar, whatever their actual dimens... turbulence sets in at the same constant value of Reynold's number in all cases of liquid flow, ... flow will be steady and streamline in each individual case, until this number is not exceeded. ... exceeding this number, the flow becomes turbulent. Even though the values of *r*, ρ and η may all ... from each other, but so long as *k* remains the same, the liquid flow will be similar in all the cas...

2.3 POISEUILLE'S FORMULA FOR THE FLOW OF A LIQUID THROUGH A CAPILLARY TUBE

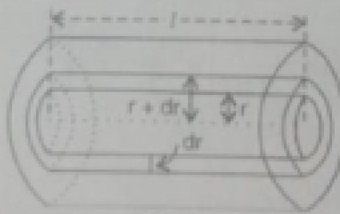


Fig. 2.3(*a*)

Fig. 2.3(*b*)

Suppose a constant pressure difference *p* is maintained between the two ends of the capillary tu... of length *l* and radius *a*, as shown in Fig. 2.3 (*a*). Consider the steady flow of a liquid of coeffici... of viscosity η through the tube. The velocity of the liquid is a maximum along the axis and is ze... at the walls of the tube. Assume that there is no radial flow. Consider a cylindrical shell of the liqu... co-axial with the tube of inner radius *r* and outer radius *r* + *dr* [Fig. 2.3(*b*)]. Let the velocity of th... liquid on the inner surface of the shell be *v* and that on the outer surface be *v* – *dv*. (*dv*/*dr*) is th... velocity gradient.

The surface area of the shell = $A = 2\,\pi\, r\, l$.

According to Newton's law of viscous flow, the backward dragging tangential force exerte... by the outer layer on the inner layer, opposite to the direction of motion

$$F_1 = -\eta A \frac{dv}{dr} = -\eta\, 2\pi r l \frac{dv}{dr}$$

The driving force on the liquid shell, accelerating it forward

$$F_2 = p\pi r^2$$

where, p = pressure difference across the two ends of the tube and

πr^2 = Area of cross-section of the inner cylinder.

When the motion is steady,

backward dragging force (F_1) = The driving force (F_2)

$$-\eta\, 2\pi r l \frac{dv}{dr} = p\,\pi r^2 \text{ or } dv = \frac{-p}{2\eta l} r\, dr.$$

Integrating, $$v = \frac{-p}{2\eta l}\frac{r^2}{2} + C.$$

where C is a constant of integration.

When $r = a$, $v = 0$. Hence, $0 = \frac{-p}{2\eta l}\frac{a^2}{2} + C$ or $C = \frac{pa^2}{4\eta l}$

$$\therefore \quad v = \frac{p}{4\eta l}(a^2 - r^2)$$

This gives us the average velocity of the liquid flowing through the cylindrical shell. Hence the volume of the liquid that flows out per second through this shell

$$dV = \begin{pmatrix}\text{Area of cross-section of the shell} \\ \text{of radius } r \text{ and thickness } dr\end{pmatrix} \times \text{Velocity of flow}$$

$$= 2\pi r\, dr \frac{p}{4\eta l}(a^2 - r^2) = \frac{\pi p}{2\eta l}(a^2 r - r^3)\, dr$$

The volume of the liquid that flows out per second is obtained by integrating the expr... for dV between the limits $r = 0$ to $r = a$.

$$V = \int_0^a \frac{\pi p}{2\eta l}(a^2 r - r^3)\, dr = \frac{\pi p}{2\eta l}\left[a^2\frac{r^2}{2} - \frac{r^4}{4}\right]_0^a$$

$$= \frac{\pi p}{2\eta l}\frac{a^4}{4}$$

or $$V = \frac{\pi p a^4}{8\eta l}$$

.4. CORRECTIONS TO POISEUILLE'S FORMULA

Two important corrections are to be applied in the Poiseuille's equation :

(*i*) **Correction for pressure head :** The outgoing liquid acquires K.E. due to its veloci... ...assing through the tube. Hence the pressure-head maintained is utilized not only for over... ...iscous resistance but also in imparting considerable K.E. to emergent liquid. So the effective p... ...s less and is given by

But

$$v = \frac{p}{4\eta l}(a^2 - r^2)$$

$\therefore$

$$E' = \pi\rho \int_0^a r \left(\frac{p}{4\eta l}\right)^3 (a^2 - r^2)^3 \, dr$$

$$= \pi\rho \left(\frac{p}{4\eta l}\right)^3 \frac{a^8}{8} = \left(\frac{\pi p a^4}{8\eta l}\right)^3 \frac{\rho}{\pi^2 a^4}$$

$$= \frac{V^3 \rho}{\pi^2 a^4}$$

The work done in overcoming viscosity is $p_1 V$ whereas total work done per unit volume is pV. Here p_1 is the effective pressure.

$\therefore$

$$pV = p_1 V + \frac{V^3 \rho}{\pi^2 a^4}$$

or

$$p_1 = p - \frac{V^2 \rho}{\pi^2 a^4}$$

$\therefore$

$$p_1 = g\rho\left(h - \frac{V^2}{\pi^2 a^4 g}\right)$$

Thus $[V^2/(\pi^2 a^4 g)]$ is the correction factor to the pressure head for gain of kinetic energy by the emergent liquid.

(*ii*) **Correction for length of tube:** At the inlet end of the tube, the flow of the liquid is not streamline for some distance. Consequently the liquid is accelerated. The effective length of the tube is thus increased from l to $l + 1.64\,a$. Thus, the corrected relation for η becomes

$$\eta = \frac{\pi a^4}{8V(l + 1.64\,a)}\left(h - \frac{V^2}{\pi^2 a^4 g}\right) g\rho$$

2.5. POISEUILLE'S METHOD FOR DETERMINING COEFFICIENT OF VISCOSITY OF A LIQUID

The liquid is taken in the constant level tank up to a height h (Fig. 2.4). A capillary tube AB is fixed to the bottom of the tank. A weighed beaker is placed below the free end B of the capillary tube. The mass m of the liquid collected in it in time t is found out.

Volume of liquid flowing per second $= V = m/(\rho \,.\, t)$ where ρ is the density of the liquid. The length l of the capillary tube is measured by a metre rod. The radius a of the capillary tube is determined very accurately, using the travelling microscope. Then from the relation

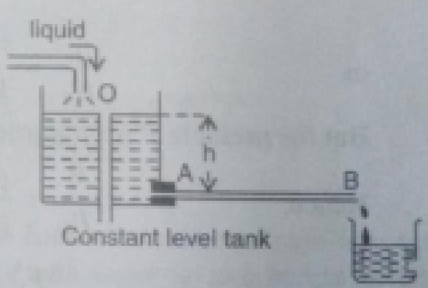


Fig. 2.4

$$\eta = \frac{\pi\, p a^4}{8 V l} \qquad \text{(where } p = h\rho g\text{)}$$

the value of η for the liquid can be easily calculated.

Comparison of viscosities : The liquid whose viscosity is η_1 is first used in the constant level tank and the volume of liquid flowing per second $= V_1 = m_1/\rho_1 \cdot t$ is determined with a capillary tube. The tube is then taken out and cleaned well. The experiment is repeated for the other liquid whose viscosity is η_2 and the volume of liquid flowing per second $= V_2 = m_2/\rho_2 \cdot t$ is determined for the same pressure head and with the same capillary tube. If l is the length of the tube, a its radius and ρ_1 and ρ_2 the densities of the two liquids,

$$\eta_1 = \frac{\pi h \rho_1 g a^4}{8 V_1 l} \text{ and } \eta_2 = \frac{\pi h \rho_2 g a^4}{8 V_2 l} \quad \therefore \quad \frac{\eta_1}{\eta_2} = \frac{\rho_1 V_2}{\rho_2 V_1}$$

ρ_1/ρ_2 can be determined with a Hare's apparatus. Thus the viscosities of two liquids can be compared.

2.6 OSTWALD'S VISCOMETER

This instrument is used to compare the viscosities of two liquids. It is also used to study the variation of viscosity of a liquid with temperature.

The apparatus consists of two glass bulbs A and B joined by a capillary tube DE bent into a U-form (Fig. 2.5). The bulb A is connected to a funnel F. The bulb B is connected to an exhaust pump through a stop-cock S. K, L, and M are fixed marks, as shown in the figure. The whole apparatus is placed inside a constant temperature bath.

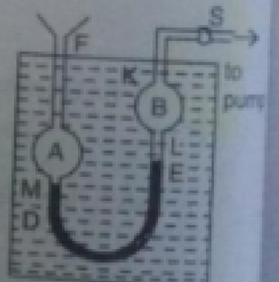


Fig. 2.5

The liquid is then introduced into the apparatus through the funnel and its volume is adjusted, so that the liquid occupies the portion between the marks K and M, when the stop-cock is closed. The stop-cock is now opened and with the help of the exhaust pump the liquid is sucked up above the mark K. The stop-cock is closed and the exhaust pump is removed. The stop-cock is again opened. The liquid is allowed to flow through the capillary tube.

The time (t_1) taken by the liquid to fall from the mark K to the mark L is noted. The experiment is then repeated with the second liquid and the time (t_2) taken by it to fall from K to L is noted.

Theory : Let η_1 and η_2 be the coefficients of viscosity and ρ_1 and ρ_2 the densities of the two liquids respectively. Let the volume of liquid between K and L be V. Then,

the rate of flow of the first liquid $= V_1 = V/t_1$

and the rate of flow of the second liquid $= V_2 = V/t_2$

Now, $\eta_1 = \dfrac{\pi \cdot P_1 \cdot a^4}{8 V_1 \cdot l}$ ...(1)

and $\eta_2 = \dfrac{\pi \cdot P_2 \cdot a^4}{8 V_2 \cdot l}$ ...(2)

or

$$\frac{\eta_1}{\eta_2} = \frac{V_2}{V_1} \times \frac{P_1}{P_2}$$

But the pressure P is proportional to the density of the liquid used ($P = h\rho g$)

Hence,

$$\frac{P_1}{P_2} = \frac{\rho_1}{\rho_2} \qquad \text{...(3)}$$

Also, dividing (2) by (1), $\dfrac{V_2}{V_1} = \dfrac{t_1}{t_2}$ ...(4)